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by
THEODORE N. EDELBAUM, LESTER L. SACKETT,
and
HARVEY L. MALCHOW
The Charles Stark Draper Laboratory, Incorporated
Cambridge, Massachusetts

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Theodore N. Edelbaum**, Lester L. Sackett† and Harvey L. Malchow
The Charles Stark Draper Laboratory, Inc.
Cambridge, Massachusetts

Abstract

A computer code which will rapidly calculate time-optimal low thrust transfers is being developed as a mission analysis tool. The final program will apply to NEP or SEP missions and will include a variety of environmental effects. The current program assumes constant acceleration. The oblateness effect and shadowing may be included. Detailed state and costate equations are given for the thrust effect, oblateness effect, and shadowing. A simple but adequate model yields analytical formulas for power degradation due to the Van Allen radiation belts for SEP missions. The program avoids the classical singularities by the use of equinoctial orbital elements. Kryloff-Bogoliuboff averaging is used to facilitate rapid calculation. Results for selected cases using the current program are given.

Introduction

This paper discusses the current version of a mission analysis computer program being developed for Goddard Space Flight Center. The program rapidly calculates time-optimal low thrust transfers between any two geocentric orbits in the presence of a strong gravitational field. The final program will encompass both nuclear or solar electric powered transfer and will also consider the effect of one high thrust impulse. Several environmental effects will be considered including shadowing, oblateness, and the solar cell power degradation due to the Van Allen radiation belts.

The code for the simplified constant acceleration problem has been completed. This code includes the effects of earth oblateness and the shadow effect. A subroutine calculates the times of entrance and exit from earth's shadow for a given orbit and the thrust is set to zero while the spacecraft is in shadow.

The application of optimal control theory yields a two point boundary value problem which is solved using a modified Newton-Raphson iteration. The new code has two distinctions. The singularities that can occur when the eccentricity is zero or the inclination is zero, and when classical orbital elements are used, are eliminated by the use of equinoctial orbital elements.[1] Kryloff-Bogoliuboff averaging [2] is used to insure rapid calculations of trajectories. Averaged orbital elements yield a first approximation to the actual elements. Five orbital elements vary slowly over several orbits, and the sixth, corresponding to position in an orbit, is eliminated by the averaging. Averaging over a single orbit is performed by quadrature. The differential equations for the approximate state and costate are solved numerically using a time step equal to several orbital revolutions.

Cefola [1] has derived the variation of parameter equations and the perturbation due to oblateness in terms of equinoctial orbital elements, but not the adjoint equations which are used here. The method of averaging has been used extensively, however, not for the present problem. Edelbaum [3, 4] has used averaging to calculate analytic solutions for special cases of optimal low thrust trajectories, and others have used averaging when considering effects such as oblateness, third body effects, and nonoptimal thrusting.[1, 5, 6]

In summary, the current version of the code calculates the constant acceleration low thrust time optimal geocentric transfer between elliptical orbits and 1) includes the effects of oblateness and shadowing, 2) avoids the classical singularities, 3) is rapid due to averaging, and 4) is general, since it calculates the optimal transfer between any initial orbit and any final orbit or a subset of the final orbital elements. Previous simulations have not combined all these qualities.

In the following section, a brief summary of Kryloff-Bogoliuboff averaging as applied to a trajectory optimization problem is presented. In succeeding sections a description of the equinoctial orbital elements is given, followed by a description of the optimization problem that we are considering. The mathematical equations for a five dimensional state consisting of the orbital elements are presented, followed by the equations needed to include the effects of shadowing and of oblateness. The equations for a seven dimensional state, including mass and accumulated particle flux (of which power is a function) are also given. Next is a description of the computer program at its current stage of development, followed by a discussion of the modeling of the effects of Van Allen radiation on the power output of the solar cells. Finally some representative examples of numerical results produced by the early version of the computer code are presented.

A typical transfer is one starting in a low altitude, eccentric, inclined orbit and ending at circular geosynchronous orbit. The results show trajectories and ΔV's for the examples of transfers.

Averaging

A great savings in computer time can be effected by considering a first approximation to the state and costate. Short period variations in the state and costate are eliminated by the averaging technique. Let the state include five orbital elements indicating the size, orientation and shape of the orbit, but not the position of the spacecraft in the orbit. Assume that these elements vary slowly over one orbit. The state may also include other quantities which vary slightly over one orbit.

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** Associate Fellow AIAA
† Member AIAA

The averaged Hamiltonian can be defined as

$$\tilde{H} = \frac{1}{T} \int_{t-\frac{T}{2}}^{t+\frac{T}{2}} H \, dt$$

where H is the unaveraged Hamiltonian and T is the orbital period. When calculating this integral the state and costate are held fixed. The motion of the spacecraft is assumed to vary in a manner described by Kepler's equation over the averaging integration. The first approximate state and costate satisfy the Euler-Lagrange equations

$$\dot{\bar{\underline{x}}}^T = \frac{\partial \tilde{H}}{\partial \bar{\underline{\lambda}}} \tag{1}$$

$$\dot{\bar{\underline{\lambda}}}^T = - \frac{\partial \tilde{H}}{\partial \bar{\underline{x}}} \tag{2}$$

where the overbar indicates the approximate quantities.

In some cases the averaging integral can be solved analytically; otherwise a numerical quadrature formula may be used. The differential equations can then be solved numerically using a time step which is much larger than but unrelated to the number of orbital revolutions.

Equinoctial Orbital Elements

By using equinoctial orbital elements the singularities that occur for zero eccentricity or inclinations of zero or ninety degrees when using classical orbital elements are avoided. (For inclinations near 180° retrograde equinoctial orbital elements can be used, although we will not consider that case in this paper.) The formulas given in this section are taken from Cefola(1).

The direct equinoctial orbital elements are defined in terms of the classical orbital elements, a, e, i, Ω, and ω by the formulas

$$\begin{aligned} a &= a \\ h &= e \sin(\omega + \Omega) \\ k &= e \cos(\omega + \Omega) \\ p &= \tan(\tfrac{i}{2}) \sin \Omega \\ q &= \tan(\tfrac{i}{2}) \cos \Omega \end{aligned} \tag{3}$$

In Cefola (1) the sixth orbital element is the mean longitude at epoch. In this paper we will consider the eccentric longitude, F, as the sixth element, defined by

$$F = E + \omega + \Omega \tag{4}$$

where E is the eccentric anomaly. This element will be eliminated from the dynamical equations by the averaging process.

The equinoctial coordinate frame is defined by the basis vectors $\hat{f}$, $\hat{g}$, $\hat{w}$, which are given below with respect to an earth equatorial coordinate frame.

$$\underline{\hat{f}} = \frac{1}{1 + p^2 + q^2} \begin{bmatrix} 1 - p^2 + q^2 \\ 2pq \\ -2p \end{bmatrix} \tag{5}$$

$$\underline{\hat{g}} = \frac{1}{1 + p^2 + q^2} \begin{bmatrix} 2pq \\ 1 + p^2 - q^2 \\ 2q \end{bmatrix} \tag{6}$$

$$\underline{\hat{w}} = \frac{1}{1 + p^2 + q^2} \begin{bmatrix} 2p \\ -2q \\ 1 - p^2 - q^2 \end{bmatrix} \tag{7}$$

This coordinate frame is illustrated in Figure 1 where $\underline{\hat{w}}$ is normal to the orbital plane.

The spacecraft position and velocity are given by

$$\underline{r} = X_1 \underline{\hat{f}} + Y_1 \underline{\hat{g}} \tag{8}$$

$$\dot{\underline{r}} = \dot{X}_1 \underline{\hat{f}} + \dot{Y}_1 \underline{\hat{g}} \tag{9}$$

where

$$X_1 = a[(1-h^2\beta)\cos F + hk\beta \sin F - k] \tag{10}$$

$$Y_1 = a[(1-k^2\beta)\sin F + hk\beta \cos F - h] \tag{11}$$

$$\dot{X}_1 = \frac{na^2}{r}[hk\beta \cos F - (1-h^2\beta)\sin F] \tag{12}$$

$$\dot{Y}_1 = \frac{na^2}{r}[(1-k^2\beta)\cos F - hk\beta \sin F] \tag{13}$$

and

$$\beta = \frac{1}{1 + \sqrt{1 - h^2 - k^2}} \tag{14}$$

$$n = \sqrt{\frac{\mu}{a^3}} \tag{15}$$

$$\frac{r}{a} = 1 - k \cos F - h \sin F \tag{16}$$

μ is the earth gravitational constant. Kepler's equation is given by

$$M + \omega + \Omega = F - k \sin F + h \cos F \tag{17}$$

The Optimization Problem

Let x represent the state (which for the NEP case includes the five orbital elements and for the SEP case includes the five orbital elements, mass, and accumulated particle flux, of which power is a function) and λ the costate. Orbital transfer time is to be minimized. The initial state is assumed to be specified and all or some combination of the final orbit elements are specified.

The differential equation for the state can be written as

$$\dot{\underline{x}} = \underline{g}_1(\underline{x}, F, t) + a(\underline{x}, t)\, G_2(\underline{x}, F, t)\, \hat{\underline{u}} \tag{18}$$

where $a(x, t)$ represents the magnitude of the thrust acceleration which can be written as a function of time in the NEP case or as a function of mass and accumulated flux (or power) in the SEP case; $\hat{\underline{u}}$ is the thrust direction;

$$G_2(\underline{x}, F) = \frac{\partial \underline{x}}{\partial \dot{\underline{r}}} \tag{19}$$

$g_1(x, F, t)$ includes all other effects not dependent on thrust direction such as oblateness and the derivatives of mass and flux.

The Hamiltonian is given by

$$H = \underline{\lambda}^T \dot{\underline{x}} \tag{20}$$

This is maximized by setting

$$\hat{\underline{u}} = \frac{G_2^T(\underline{x}, F)\underline{\lambda}}{|G_2^T(\underline{x}, F)\underline{\lambda}|} \tag{21}$$

The maximum value of H, denoted by H*, is then given by

$$H^* = \underline{\lambda}^T \underline{g}_1 + a|G_2^T \underline{\lambda}| \tag{22}$$

The method of averaging may now be used to determine the first order approximation to the mean motion of the system. The averaged Hamiltonian, $\tilde{H}$, is defined by

$$\tilde{H} = \frac{1}{T} \int_{t-\frac{T}{2}}^{t+\frac{T}{2}} H^*(\overline{\underline{x}}, \overline{\underline{\lambda}}, \overline{t}, F(t))\, dt \tag{23}$$

where T is the period of the orbit and where the five orbital elements, their adjoints, and any explicit time dependence not involving the motion of the spacecraft in its orbit (indicated by $\overline{t}$) are held constant over the averaging interval. (For example, in the five dimensional, NEP, constant thrust case, the thrust acceleration, which is an explicit function of time, is held constant over the interval. Also for the SEP case, the sun's direction, a function of time, is held constant when calculating the shadow location.) Averaging with respect to the eccentric longitude F, $\tilde{H}$ is given by

$$\tilde{H} = \frac{1}{T} \int_{-\pi}^{\pi} H^*(\overline{\underline{x}}, \overline{\underline{\lambda}}, \overline{t}, F)\left(\frac{dt}{dF}\right) dF \tag{24}$$

where

$$\frac{dt}{dF} = \frac{T}{2\pi}(1 - \overline{k}\cos F - \overline{h}\sin F) \tag{25}$$

obtained from Kepler's equation. Let

$$s(\overline{h}, \overline{k}, F) = \frac{1}{T}\frac{dt}{dF} \tag{26}$$

Then the Euler-Lagrange equations for this system are

$$\dot{\overline{\underline{x}}} = \frac{\partial \tilde{H}^T}{\partial \overline{\underline{\lambda}}} = \int_{-\pi}^{\pi} \dot{\underline{x}}(\overline{\underline{x}}, \overline{\underline{\lambda}}, \overline{t}, F)\; s(\overline{h}, \overline{k}, F)\, dF \tag{27}$$

$$\dot{\overline{\underline{\lambda}}}^T = -\frac{\partial \tilde{H}}{\partial \overline{\underline{x}}} = -\frac{\partial}{\partial \overline{\underline{x}}} \int_{-\pi}^{\pi} \overline{\underline{\lambda}}^T \dot{\underline{x}}(\overline{\underline{x}}, \overline{\underline{\lambda}}, \overline{t}, F)\, s(\overline{h}, \overline{k}, F)\, dF \tag{28}$$

Since we are minimizing time, $\tilde{H}(t_f)$ must equal one. Transversality conditions and conditions on the final orbital elements yield five (or seven) additional final conditions. The two point boundary value problem can be solved using a modified Newton-Raphson method by iterating on the initial costate and the final time, t_f, in order to meet the final conditions. In our program the partial derivatives of the final conditions with respect to the initial costate are calculated numerically.

Note that the approximate thrust direction in the equinoctial coordinate frame on a particular orbit can be obtained by substituting the first approximate state and costate into Equation (21)

$$\overline{\hat{\underline{u}}} = \frac{G_2^T(\overline{\underline{x}}, F)\overline{\underline{\lambda}}}{|G_2^T(\overline{\underline{x}}, F)\overline{\underline{\lambda}}|} \tag{29}$$

Equations for a Five Dimensional State

In this section the equations needed for the five dimensional NEP case will be given. Only the effect of thrusting will be considered. Many of these equations will be applicable to the seven dimensional SEP case which will be described later. To avoid confusion with the seven dimensional state, let $\underline{z}$ be the vector of the five orbital elements,

The unaveraged variation of parameters equation is given by

$$\dot{\underline{z}} = a(t)\, M(\underline{z}, F)\frac{M^T(\underline{z}, F)\underline{\lambda}}{|M^T(\underline{z}, F)\underline{\lambda}|} \tag{30}$$

For constant acceleration

$$a(t) = a_0 \tag{31}$$

and for constant thrust

$$a(t) = \frac{\alpha c}{m_0 - \alpha t} \tag{32}$$

If the power decays exponentially

$$a(t) = \frac{a_0 e^{-bt}}{1 + \frac{a_0}{bc}(e^{-bt} - 1)} \tag{33}$$

Here α is the mass flow rate

$$\dot{m} = -\alpha, \tag{34}$$

b is defined by

$$\dot{P} = -bP \tag{35}$$

and c is the jet velocity. The initial acceleration is defined by

$$a_0 = \frac{2P_0}{m_0 c} \tag{36}$$

and M is the 5 x 3 matrix

$$M(\underline{z}, F) = \frac{\partial \underline{\dot{z}}}{\partial \underline{\dot{r}}} \tag{37}$$

The elements of this matrix are listed in Table 1. In addition to the quantities defined in the section on equinoctial orbital elements, the partials of X_1 and Y_1 with respect to h and k are required. These are listed in Table 2. These partials differ from those in Cefola since we consider F as an orbital element rather than as a function of h and k; thus when partials are taken, F is held constant. This assumption also affects the appearance of the expressions given in Table 1 when compared with Table 3 of Cefola. However, if the expressions are written out in detail they are seen to be identical.

The averaged Hamiltonian is given by

$$\tilde{H}_z = a(t) \int_{-\pi}^{\pi} |M^T(\underline{\bar{z}}, F)\underline{\bar{\lambda}}_z| \, s(\bar{h}, \bar{k}, F) \, dF \tag{38}$$

The Euler-Lagrange equations are then

$$\dot{\underline{\bar{z}}} = a(t) \int_{-\pi}^{\pi} M(\underline{\bar{z}}, F) \frac{M^T(\underline{\bar{z}}, F)\underline{\bar{\lambda}}_z}{|M^T(\underline{\bar{z}}, F)\underline{\bar{\lambda}}_z|} s(\bar{h}, \bar{k}, F) dF \tag{39}$$

and

$$\dot{\underline{\bar{\lambda}}}_z^T = -\int_{-\pi}^{\pi} \left\{ \underline{\bar{\lambda}}_z^T \frac{\partial \dot{\underline{z}}}{\partial \underline{\bar{z}}} (\underline{\bar{z}}, \underline{\lambda}, F) \, s(\bar{h}, \bar{k}, F) + \underline{\bar{\lambda}}_z^T \dot{\underline{z}} \frac{\partial s(\bar{h}, \bar{k}, F)}{\partial \underline{\bar{z}}} \right\} dF \tag{40}$$

If $\bar{z}_i$ is the ith component of $\underline{\bar{z}}$, then

$$\frac{\partial \dot{\underline{z}}}{\partial \bar{z}_i} = a(t) \frac{\partial M(\underline{\bar{z}}, F)}{\partial \bar{z}_i} \frac{M^T(\underline{\bar{z}}, F)\underline{\bar{\lambda}}}{|M^T(\underline{\bar{z}}, F)\underline{\bar{\lambda}}|} \tag{41}$$

The partials of M with respect to a, h, k, p, and q are given in Tables 3 - 7. In addition the expressions for the partials of X_1 and Y_1 as well as the second partials of X_1 and Y_1 are needed and so are listed in Tables 8 and 9. Finally

$$\frac{\partial s^T}{\partial \underline{\bar{z}}} = \frac{1}{2\pi} \begin{bmatrix} 0 \\ -\sin F \\ -\cos F \\ 0 \\ 0 \end{bmatrix} \tag{42}$$

The integrals in Equations (39) and (40) can be evaluated by a quadrature formula. We have typically used a 16 point gaussian quadrature.

The Shadow Effect

For SEP missions, the thrusting will be shut off while the spacecraft is in the earth's shadow. The entry and exit angles are needed in order to perform the averaging integral. In calculating these angles the following assumptions are made. The shadow is cylindrical; the earth revolves around the sun in an elliptical orbit; and over one spacecraft revolution, the sun's direction is fixed.

State and Costate Equations

It is assumed that immediately upon entrance to the shadow, the thrust is turned off, and immediately upon exit, it is turned on. The integrals for the approximated state and costate, Equations (39) and (40), must be altered appropriately. Since the thrust acceleration is zero when the spacecraft is in shadow, Equation (32) becomes

$$\tilde{H}_z = \int_{F_1}^{F_2} a(t) \, |M^T(\underline{\bar{z}}, F)\underline{\bar{\lambda}}| \, s(\bar{h}, \bar{k}, F) \, dF \tag{43}$$

where only the limits on the integral have been changed. F_2 corresponds to the entrance angle and F_1 to the exit angle, and we assume that $F_2 > F_1$, $F_2 - F_1 < 180°$. The state differential equation is then

$$\dot{\underline{\bar{z}}} = a(t) \int_{F_1}^{F_2} M(\underline{\bar{z}}, F) \frac{M^T(\underline{\bar{z}}, F)\underline{\bar{\lambda}}}{|M^T(\underline{\bar{z}}, F)\underline{\bar{\lambda}}|} s(\bar{h}, \bar{k}, F) \, dF \tag{44}$$

Since F_2 and F_1 are functions of the orbital elements, by Leibnitz's rule,

$$\dot{\underline{\bar{\lambda}}}^T = -\frac{\partial \tilde{H}_z}{\partial \underline{\bar{z}}} = -\int_{F_1}^{F_2} \left(\underline{\bar{\lambda}}^T \frac{\partial \dot{\underline{z}}}{\partial \underline{\bar{z}}} s(\bar{h}, \bar{k}, F) + \underline{\bar{\lambda}}^T \dot{\underline{z}} \frac{\partial s(\bar{h}, \bar{k}, F)}{\partial \underline{\bar{z}}} \right) dF - \frac{dF}{d\underline{\bar{z}}} \bigg|_{F_2} a(t) |M^T(\underline{\bar{z}}, F_2)\underline{\bar{\lambda}}| + \frac{dF}{d\underline{\bar{z}}} \bigg|_{F_1} a(t) |M^T(\underline{\bar{z}}, F_1)\underline{\bar{\lambda}}| \tag{45}$$

The Shadow Equation

From geometrical considerations an equation can be derived which the entry and exit angles must satisfy. Such an equation is given in Escabal[7], and the equation given in this section is essentially the same, except that it is given in terms of equinoctial orbital elements.

The spacecraft position is given by

$$\underline{r} = X_1\hat{\underline{f}} + Y_1\hat{\underline{g}} \tag{46}$$

where X_1 and Y_1 were given in Equations (10) and (11). Let the unit vector from the earth to the sun be given by

$$\hat{\underline{R}}_s = X_s\hat{\underline{f}} + Y_s\hat{\underline{g}} + Z_s\hat{\underline{w}} \tag{47}$$

This is in terms of the equinoctial coordinate frame and thus depends on the equinoctial orbital elements p and q. If a_e designates the earth's radius, the cosine of the angle between $\underline{r}$ and $\hat{\underline{R}}$ is given by

$$\frac{\hat{\underline{R}}_s \cdot \underline{r}}{|\hat{\underline{R}}_s|\,|\underline{r}|} = -\frac{(|\underline{r}|^2 - a_e^2)^{\frac{1}{2}}}{|\underline{r}|} \tag{48}$$

or,

$$X_1X_s + Y_1Y_s = -(|\underline{r}|^2 - a_e^2)^{\frac{1}{2}} \tag{49}$$

Squaring and rearranging

$$S \equiv (1-X_s^2)X_1^2 + (1-Y_s^2)Y_1^2 - 2X_sY_sX_1Y_1 - a_e^2 = 0 \tag{50}$$

This is the shadow equation which must be satisfied by the entry and exit angles. X_1 and Y_1 are functions of cos F, sin F, a, h, and k (see Equations (10) and (11)). By further manipulations one can derive a quartic equation in cos F. The coefficients of this quartic equation are given in Table 10. Spurious roots can be eliminated by the criteria that S = 0 and that $\hat{R} \cdot r < 0$. In addition, for the entry angle $\partial S/\partial F < 0$ and for the exit angle $\partial S/\partial F > 0$.

Derivatives of F and S

The derivative of F with respect to $\overline{z}$ is needed to evaluate the costate equation. It can be obtained implicitly from the shadow equation.

$$\frac{dF}{d\underline{\overline{z}}} = \frac{\partial S}{\partial \underline{\overline{z}}} \Bigg/ \frac{\partial S}{\partial F} \tag{51}$$

These partials are listed in Table 11. Note that in calculating $\partial S/\partial p$ and $\partial S/\partial q$ we have taken into account the fact that the sun's direction is given in equinoctial coordinates.

Oblateness

In previous sections we have considered only perturbations to the inverse square motion caused by thrusting. In this section the effect of oblateness (J_2) is considered. This is an additive term to the variation of parameters formulas such as indicated by $g_1(x, F, t)$ in Equation (18). The single averaged perturbing potential due to J_2 has been calculated in terms of equinoctial coordinates by Cefola[1] and is repeated here in Table 12. R_e is the equatorial radius of the earth and $J_2 = .0010827$. These formulas enter the averaged Hamiltonian as coefficients of the costate (outside the integral since the averaging effect has already been accounted for).

If $\dot{\underline{\overline{z}}}_a$ indicates the perturbation due to thrust as given in Equation (39), then the Hamiltonian is given by

$$\tilde{H} = \underline{\lambda}^T\dot{\underline{\overline{z}}}_{J_2} + \underline{\lambda}^T\dot{\underline{\overline{z}}}_a \tag{52}$$

The state equation is

$$\dot{\underline{\overline{z}}} = \dot{\underline{\overline{z}}}_{J_2} + \dot{\underline{\overline{z}}}_a \tag{53}$$

The costate equation is

$$\dot{\underline{\overline{\lambda}}}^T = -\frac{\partial \tilde{H}_z}{\partial \underline{\overline{z}}} = -\underline{\overline{\lambda}}^T \frac{\partial \dot{\underline{\overline{z}}}_{J_2}}{\partial \underline{\overline{z}}} - \int_{-\pi}^{\pi} \left\{ \underline{\overline{\lambda}}^T \frac{\partial \dot{\underline{z}}_a}{\partial \underline{\overline{z}}} s + \underline{\overline{\lambda}}^T \dot{\underline{z}}_a \frac{\partial s}{\partial \underline{\overline{z}}} \right\} dF \tag{54}$$

The partials indicated by $\partial\dot{\overline{z}}_{J_2}/\partial\overline{z}$ in the above expression are given in Tables 13-17.

Equations for a Seven Dimensional State

Up to this point we have assumed a five dimensional state consisting of the five orbital elements a, h, k, p, and q. Thrust was not a function of the state (except in the case of shadowing when the entry and exit times were a function of the orbital elements). For SEP missions the solar cell performance will degrade in the presence of Van Allen radiation. This will cause the amount of power delivered to the thrusters to decrease with time. The power degradation can be modeled as a function of the amount of equivalent electron flux intercepted by the spacecraft solar cells. The accumulated flux is dependent on the trajectory as well as time (through the earth's rotation). The modelling of the Van Allen radiation and its effect on the solar cells is discussed in a later section. Power delivered to the thrusters is also influenced by the varying distance to the sun as a result of the ellipticity of the earth's orbit. In this section, we will simply assume that we have analytic expressions given for power as a function of accumulated flux (and time) and for the flux rate.

$$\begin{aligned} P &= P(N, t) \\ \dot{N} &= f(\underline{z}, F, t) \end{aligned} \tag{55}$$

P is assumed to be zero when the spacecraft is in shadow.

For this study, it is assumed that thrust level is proportional to input power, i.e. thrust is a function only of beam current with specific impulse and efficiency constant. Thrust acceleration is given by

$$a = \frac{2P}{mc} \tag{56}$$

and mass flow rate by

$$\dot{m} = -\frac{2P}{c^2} \tag{57}$$

Thus we can consider a seven dimensional state of five orbital elements, mass, and accumulated particle flux.

$$\underline{x} = \begin{bmatrix} \underline{z} \\ m \\ N \end{bmatrix} \tag{58}$$

Since m and N are varying slowly the first approximation of these quantities as well as the orbital elements can be considered.

In the remainder of this section the state and costate derivative equations will be given. Only thrusting in an inverse square field will be considered. The oblateness effect could easily be included as in the five dimensional case.

The averaged Hamiltonian is given by

$$\begin{aligned}\tilde{H} &= \int_{-\pi}^{\pi} \overline{\lambda}^T \dot{\underline{x}}(\overline{\underline{x}}, F, \overline{t})\, s(\overline{h}, \overline{k}, F)\, dF \\ &= \int_{F_1}^{F_2} \left\{ \frac{2P(\overline{N}, \overline{t})}{\overline{m}c} \left| M^T(\overline{\underline{z}}, F)\overline{\underline{\lambda}}_z \right| \right. \\ &\quad \left. - \overline{\lambda}_m \frac{2P(\overline{N}, \overline{t})}{c^2} \right\} s(\overline{h}, \overline{k}, F)\, dF \\ &\quad + \int_{-\pi}^{\pi} \overline{\lambda}_N f(\overline{\underline{z}}, F, \overline{t})\, s(\overline{h}, \overline{k}, F)\, dF \end{aligned} \tag{59}$$

The approximate state must satisfy:

$$\begin{aligned}\dot{\overline{\underline{z}}} &= \frac{2P(\overline{N}, t)}{\overline{m}c} \int_{F_1}^{F_2} M(\overline{\underline{z}}, F) \frac{M^T(\overline{\underline{z}}, F)\overline{\underline{\lambda}}_z}{\left| M^T(\overline{\underline{z}}, F)\overline{\underline{\lambda}}_z \right|}\, s(\overline{h}, \overline{k}, F)\, dF \\ \dot{\overline{m}} &= -\frac{2P(\overline{N}, t)}{c^2} \frac{T_L}{T}, \\ \dot{\overline{N}} &= \int_{-\pi}^{\pi} f(\overline{\underline{z}}, F, \overline{t})\, s(\overline{h}, \overline{k}, F)\, dF, \end{aligned} \tag{60}$$

where T_L is the time spent in sunlight, T is the orbital period, F_2 and F_1 are the shadow entry and exit angles respectively.

The approximate costate must satisfy

$$\begin{aligned}\dot{\overline{\underline{\lambda}}}_z^T &= -\int_{F_1}^{F_2} \left\{ \overline{\underline{\lambda}}_z^T \frac{\partial \dot{\underline{z}}}{\partial \underline{z}} (\overline{\underline{x}}, F, \overline{t})\, s(\overline{h}, \overline{k}, F) \right. \\ &\quad \left. + \left(\overline{\underline{\lambda}}_z^T \dot{\underline{z}}(\overline{\underline{x}}, F, \overline{t}) + \overline{\lambda}_m \dot{m}(\overline{N}, t) \right) \frac{\partial s(\overline{h}, \overline{k}, F)}{\partial \underline{z}} \right\} dF \\ &\quad - \left. \frac{dF}{d\underline{z}} \right|_{F_2} \left[\overline{\underline{\lambda}}_z^T \dot{\underline{z}}(\overline{\underline{x}}, F_2, t) + \overline{\lambda}_m \dot{m} \right] s(\overline{h}, \overline{k}, F_2) \\ &\quad + \left. \frac{dF}{d\underline{z}} \right|_{F_1} \left[\overline{\underline{\lambda}}_z^T \dot{\underline{z}}(\overline{\underline{x}}, F_1, t) + \overline{\lambda}_m \dot{m} \right] s(\overline{h}, \overline{k}, F_1) \\ &\quad - \int_{-\pi}^{\pi} \left\{ \overline{\lambda}_N \frac{\partial \dot{N}}{\partial \underline{z}} (\overline{\underline{z}}, F, t)\, s(\overline{h}, \overline{k}, F) \right. \\ &\quad \left. + \overline{\lambda}_N \dot{N}(\overline{\underline{z}}, F, \overline{t}) \frac{\partial s}{\partial \underline{z}} (\overline{h}, \overline{k}, F) \right\} dF \\ \dot{\overline{\lambda}}_m &= \frac{2P(\overline{N}, t)}{\overline{m}^2 c} \int_{F_1}^{F_2} \left| M^T(\overline{\underline{z}}, F)\overline{\underline{\lambda}}_z \right| s(\overline{h}, \overline{k}, F)\, dF \\ \dot{\overline{\lambda}}_N &= -\frac{\partial P(\overline{N}, t)}{\partial \overline{N}} \frac{2}{\overline{m}c} \int_{F_1}^{F_2} \left\{ \left| M^T(\overline{\underline{z}}, F)\overline{\underline{\lambda}}_z \right| - \overline{\lambda}_m \frac{2}{c^2} \right\} \times \\ &\quad s(\overline{h}, \overline{k}, F)\, dF \end{aligned} \tag{61}$$

These expressions can be simplified somewhat using Equation (42) and by denoting

$$\begin{aligned} H_z &= \underline{\lambda}_z \dot{\underline{z}}, \\ H_m &= \lambda_m \dot{m}, \\ H_N &= \lambda_N \dot{N} \end{aligned} \tag{62}$$

where

$$H = H_z + H_m + H_N,$$

so that

$$\tilde{H}_z = \int_{F_1}^{F_2} \underline{\bar{\lambda}}_z^T \dot{\underline{z}}(\bar{x}, F, \bar{t})\, s(\bar{h}, \bar{k}, F)\, dF$$

$$\tilde{H}_m = -\bar{\lambda}_m \frac{2}{c^2} P(\bar{N}, t) \frac{T_L}{T} \tag{63}$$

$$\tilde{H}_N = \bar{\lambda}_N \int_{-\pi}^{\pi} \dot{N}(\bar{z}, F, \bar{t})\, s(\bar{h}, \bar{k}, F)\, dF$$

Then

$$\dot{\underline{\bar{\lambda}}}_z{}^T = -\int_{F_1}^{F_2} \left\{ \underline{\bar{\lambda}}_z{}^T \frac{\partial \dot{\underline{z}}}{\partial \underline{\bar{z}}} s + H_z \frac{\partial s}{\partial \underline{\bar{z}}} \right\} dF$$

$$+ \bar{\lambda}_m \frac{2P(\bar{N}, t)}{c^2} \frac{T}{2\pi} \begin{bmatrix} 0 \\ \cos F_2 - \cos F_1 \\ -\sin F_2 + \sin F_1 \\ 0 \\ 0 \end{bmatrix}^T \tag{64}$$

$$\frac{-dF}{d\underline{z}} (H_z + H_m) s \Bigg|_{F_1}^{F_2} - \int_{-\pi}^{\pi} \left\{ \bar{\lambda}_N \frac{\partial \dot{N}}{\partial \underline{\bar{z}}} s \right.$$

$$\left. + \bar{\lambda}_N \dot{N} \frac{\partial s}{\partial \underline{\bar{z}}} \right\} dF$$

$$\dot{\bar{\lambda}}_m = \tilde{H}_z / \bar{m}$$

$$\bar{\lambda}_N = -\frac{\partial P(\bar{N}, t)}{\partial \bar{N}} \left[\tilde{H}_z + \tilde{H}_m \right] \frac{1}{P(\bar{N}, t)}$$

Note that $\partial \dot{z}/\partial \bar{z}$ was given by Equation (41) but with $a(t)$ replaced by $2P(\bar{N})/\bar{m}c$ and $dF/d\underline{\bar{z}}$ by Equation (51).

Organization of the Computer Code

The computer program has an outer loop for the solution of the two point boundary value problem. Estimates of the initial costate and final time are read in. A modified Newton-Raphson method is used to iterate on the final conditions. Individual trajectories are evaluated by calling a prediction-corrector differential equation solving subroutine. The differential equation routine calls the function evaluation subroutine which in turn calls a quadrature subroutine to evaluate the quantities which must be numerically averaged (such as thrust effect). The function routine also calls other subroutines to find the shadow entry and exit angles and to evaluate analytically averaged functions (such as due to oblateness). The final converged values for initial costate and final time are printed and the time history of the extremal trajectory is also printed.

Various two point boundary value problem iterator routines, differential equation solution subroutines, or quadrature subroutines may be used. In the program's current stage of development constant acceleration is assumed. Oblateness and shadowing may be included. Final condition options include all five orbital elements specified or a, e, and i specified and Ω and ω free. Either a 4, 8, 16, or 32 point gaussian quadrature subroutine may be used. Some examples of results using this program are given in a later section.

Solar Cell Degradation Model

Part of the low-thrust geocentric transfer problem involves a consideration of the effects of particulate radiation in the Van Allen belts. The radiation belts have a well-defined spatial structure, and hence SEP missions must consider the position-dependent degradation of solar cells as an important element in orbit shaping.

For use with the optimization code it was required to construct a solar cell degradation model that was compatible with the overall analytical framework of the orbit-raising problem. To satisfy this requirement the model was made analytical, that is, defined by analytic functions on the entire space of the related variables. This approach also yields a model whose run time is reasonable in relation to the other problems being solved.

In this section are given the analytic expressions for the spatially dependent damaging particle flux, $n(r(t))$, and for thruster power as a function of the total degrading particle fluence, $P(N)$, where $\dot{N} = n(\underline{r}(t))$.

There are a number of alternate computational approaches to the problem of obtaining the quantities $n(r(t))$ and $P(N)$. The approach taken here is to construct a geometric space of equivalent 1MEV electron flux. Each geometric point in this space is associated analytically with a flux value which provides the required $\dot{N}$ value.

The model assumes n-on-p type silicon cells with 10 ohm-cm base resistivity. A cover shield thickness of 6 mils is assumed with semi-infinite back shielding.

The particle field is assumed to be azimuth independent. At a point R, λ in geomagnetic coordinates, where R is radius expressed in earth radii and λ is the geomagnetic latitude, the equivalent 1MEV electron flux is modeled by the function

$$\dot{N}(R, \lambda) = A_0 \exp\left\{ \sum_{i=1}^{9} A_i f_i(R, \lambda) \right\} \tag{65}$$

where the functions $f_i(R, \lambda)$ are given by

$f_1 = u \cdot w$	$f_4 = u \cdot v$	$f_7 = \sqrt{u} \cdot w$
$f_2 = u^2 \cdot w$	$f_5 = u^3 v^2$	$f_8 = u^{.25} \cdot w$
$f_3 = u^2 \cdot w \cdot v$	$f_6 = u^3 v^3$	$f_9 = u^{.2} \cdot w$

with

$u = 1/R,\ v = \sin\lambda,\ w = \cos\lambda.$

The constant coefficients A_i are:

$$A_1 = .720 \times 10^3 \qquad A_5 = .905 \times 10^2$$
$$A_2 = -.140 \times 10^3 \qquad A_6 = -.916 \times 10^2$$
$$A_3 = -.183 \times 10^2 \qquad A_7 = -.227 \times 10^4$$
$$A_4 = .206 \times 10^2 \qquad A_8 = .597 \times 10^4$$
$$A_9 = -.425 \times 10^4$$

and

$$A_0 = .222 \times 10^5$$

The analytical function $\dot{N}$ was derived by the application of an IBM multivariate regression code to a sample of the field data points. The resulting function has continuous partial derivatives in R and λ and exhibits the proper monotonicity in the outer parts of the field. The R coordinate is sampled at $1R_e$ intervals from $1.2R_e$ out to the last increment which yields a flux greater than 10^5 1MEV electrons/cm^2 sec. Also the field is sampled at $R_e = 1.0166$ ($\approx$ 100 km altitude) to better define the low altitude orbit field. Angular sample points are chosen at 10° intervals from 0° to 60°.

To calculate the particle energy spectrum the R, λ sample points have been transformed into magnetic coordinates B (field strength) and L (integral invariant) by use of the magnetic dipole transformation:

$$L = R/\cos^2\lambda$$
$$B = (B_0/R^3)(1 + 3\sin^2\lambda)^{1/2} \tag{66}$$

These coordinates were then used as inputs to the MODEL[8] computer code which provides spectral flux information based upon actual measurements.

Conversion of the MODEL-produced spectral flux to equivalent 1MEV electrons has been accomplished by application of formulas derived from the curves of Rasmussen[9]. These formulas have been compiled in computer codes which add the MODEL-produced flux value with an appropriate weight for each R, λ point of the field.

For protons the lower energy cut-off is determined by the thickness of the coverslide which shields the solar cells. The relationship between shield thickness and low energy cut-off, based upon the data of Rasmussen, is

$$E_c = 1.53\ (\ell)^{0.66}\ [\text{MEV}] \tag{67}$$

where ℓ is the shield thickness in mils. For a shield thickness of 6 mils this formula yields a low energy cut-off of approximately 5 MEV.

The proton model is approximately independent of shield thickness once the proton energy is sufficient to penetrate the shield, thus the formula for converting protons into equivalent 1MEV electrons uses only a minimum energy shield related function. Above the energy of cut-off (5 MEV) the proton conversion formula was derived by linear fitting of Rasmussen's curves. The resulting function is

$$f_p = \begin{cases} 10^{-0.72\,\ell n\,E + 4.127} & 5 \le E < 11 \text{ MEV} \\ 2200 & 11 < E < 46 \text{ MEV} \\ 10^{-1.1\,\log_{10} E + 5.26} & 46 \le E \quad \text{MEV} \end{cases} \tag{68}$$

The equation for converting electron flux to equivalent 1MEV electrons was also derived from the curves of Rasmussen. A functional form was assumed as follows:

$$f = \exp(g(\ell)\,h(E)) \tag{69}$$

where E is energy in MEV. Both g and h were then expanded in a power series and h was fit to specific curves for selected ℓ values by regression analysis. Then the coefficients $g(\ell)$ were fit. The resulting function is

$$f_e = \exp \sum_{i=0}^{4} [(A_i + B_i \cdot (\ell))\,E^i] \tag{70}$$

with constants A_i and B_i as follows.

i	A_i	B_i
0	-5.212	-0.1645
1	6.297	0.1275
2	-2.233	-0.03198
3	0.3489	0.005410
4	-0.01954	-0.0002690

Thus all electron fluxes produced by the MODEL code were passed through the above equation to produce equivalent 1MEV electrons, and these were summed over all energy interods to yield the total equivalent 1MEV electron flux at each point R, λ.

The power loss model is given by the function

$$D(N) = \exp[-(.4364 \times 10^{-12})(\log_{10}(N))^{10}],\ N \ge 1 \tag{71}$$

where N is the cumulative 1MEV electron flux from Equation (65). This function is a least squares fit to a model of degradation vs. fluence given by Luft and Rauschenback[10]. D(N) is a degradation factor whose value ranges from 1 to 0 over the fluence range 1 to ∞. Actual power to the thrusters will be a function of the initial power output before particulate damage, distance to the sun which affects the amount of energy falling on the cells and other effects. For our model we assume that power to the thrusters is given by

$$P(N,t) = D(N)\left(\frac{R_s(t)}{R_s(t_0)}\right)^{\eta}(P(1,t_0) + P_h) - P_h \tag{72}$$

where R_s is the distance to the sun, P_h is a small housekeeping power, and η is an exponent.

For the evaluation in the trajectory optimization program of the analytic expressions for $\dot{N}$ given in Equation (65) the transformation between the equinoctial coordinate frame and geomagnetic coordinates must be used. Equations (65) and (72) are essentially those referred to in Equation (55).

Numerical Results

A number of cases have been run using the current version of the computer code which assumes constant thrust acceleration. In this section, results for a typical mission which involves a transfer from a low eccentric inclined orbit to circular geosynchronous orbit will be given. Three cases were run for this transfer; Case I assumes no oblateness or shadowing, Case II assumes oblateness but no shadowing and Case III assumes both oblateness and shadowing effects are included.

The initial orbit has a semi-major axis of 10509 km, an eccentricity of .325, an inclination of 28.5° with longitude of ascending node and argument of perigee both 0°. The final orbit has a semi-major axis of 42241.19 km with zero eccentricity and inclination. An acceleration of 10^{-4} g's (9.798×10^{-4} km/sec^2) is assumed and the transfer begins on Julian Day 2444239.0 (Jan. 0, 1980).

The thrust direction for particular orbits is of interest. In Figures 2-4 are plotted the orbit and the planar component of the unit vector indicating thrust acceleration direction for three particular orbits of the Case III trajectory, namely, the initial orbit, an intermediate orbit, and the final orbit. The intermediate orbit corresponds to a time 31.7 days into the transfer and the equinoctial orbital elements yield an inclination of 16.7° and an eccentricity of .287.

The components of the state and the costate are plotted in Figures 5-9 in equinoctial coordinates for the three cases. The ΔV is equal to the thrust acceleration magnitude times the time during which acceleration is on (it is off only in Case III during the portion of the orbit which is in shadow). For Case I, the ΔV is 4.30 km/sec, for Case II it is 4.33 km/sec, and for Case III it is 4.41 km/sec. As would be expected with the oblateness effect included, ΔV is increased slightly. When the shadow effect is also included, total transfer time is increased by about 12% but ΔV is increased only slightly for this mission.

The program is written in FORTRAN IV and these runs were done on an IBM 360/75 computer using the G compiler. A 16-point quadrature was used with time steps for the differential equation solution of about three days (for some steps this was halved). As an example, Case I converged in 14 iterations with 28 trajectory calls, the total program running in 2.6 minutes. With oblateness included, run time for a single trajectory increases about 5%, with shadowing, run time for a single trajectory is up another 35%. These figures are meant to give only a very rough idea of the performance of the program. A certain amount of coding optimization must yet be done; the H compiler would reduce run time; and a more thorough study of the relation between accuracy and time step and quadrature formula must be performed. Including the power degradation effect will increase run time.

Conclusions

In this paper, we discussed the current stage of development of a low thrust optimal satellite raising program. Through the use of equinoctial orbital elements, classical singularities are eliminated. The method of averaging allows quicker trajectory evaluation than a full numerical integration. The analytic model for the effect of solar cell degradation on power also contributes to rapid calculation of the optimal trajectory and is consistent with the other approximations (such as averaging, gravity harmonic terms limited to J_2, and cylindrical shadow assumptions). A more detailed study of run time and relative accuracies using different quadrature formulas and time steps in the differential equation evaluation will be made when the degradation model has been integrated with the optimal trajectory program. The simplified problem program has already yielded some interesting results for selected missions.

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Table 1. Elements of M

$$M_{11} = \frac{2\dot{X}_1}{n^2 a}, \quad M_{12} = \frac{2\dot{Y}_1}{n^2 a}, \quad M_{13} = 0$$

$$M_{21} = \frac{\sqrt{1-h^2-k^2}}{na^2}\left[\frac{\partial X_1}{\partial k} + \frac{\dot{X}_1}{n}(\sin F - h\beta)\right]$$

$$M_{22} = \frac{\sqrt{1-h^2-k^2}}{na^2}\left[\frac{\partial Y_1}{\partial k} + \frac{\dot{Y}_1}{n}(\sin F - h\beta)\right]$$

$$M_{23} = \frac{k(qY_1 - pX_1)}{na^2\sqrt{1-h^2-k^2}}$$

$$M_{31} = -\frac{\sqrt{1-h^2-k^2}}{na^2}\left[\frac{\partial X_1}{\partial h} - \frac{\dot{X}_1}{n}(\cos F - k\beta)\right]$$

$$M_{32} = -\frac{\sqrt{1-h^2-k^2}}{na^2}\left[\frac{\partial Y_1}{\partial h} - \frac{\dot{Y}_1}{n}(\cos F - k\beta)\right]$$

$$M_{33} = -\frac{h(qY_1 - pX_1)}{na^2\sqrt{1-h^2-k^2}}$$

$$M_{41} = 0, \quad M_{42} = 0, \quad M_{43} = \frac{(1+p^2+q^2)Y_1}{2na^2\sqrt{1-h^2-k^2}}$$

$$M_{51} = 0, \quad M_{52} = 0, \quad M_{53} = \frac{(1+p^2+q^2)X_1}{2na^2\sqrt{1-h^2-k^2}}$$

Table 2. Partials of X_1 and Y_1 with Respect to h and k

$$\frac{\partial X_1}{\partial h} = a\left[-h\beta\cos F - \frac{(\beta + h^2\beta^3)}{1-\beta}(h\cos F - k\sin F)\right]$$

$$\frac{\partial Y_1}{\partial h} = a\left[k\beta\cos F - 1 + \frac{hk\beta^3}{1-\beta}(h\cos F - k\sin F)\right]$$

$$\frac{\partial X_1}{\partial k} = a\left[h\beta\sin F - 1 - \frac{hk\beta^3}{1-\beta}(h\cos F - k\sin F)\right]$$

$$\frac{\partial Y_1}{\partial k} = a\left[-k\beta\sin F + \frac{(\beta + k^2\beta^3)}{1-\beta}(h\cos F - k\sin F)\right]$$

Table 3. Partial of M with Respect to a

$$\frac{\partial M}{\partial a} = \frac{1}{2a}\begin{bmatrix} 3 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} M$$

Table 4. Partial of M with Respect to h

$$\frac{\partial M_{11}}{\partial h} = \frac{2}{n^2 a}\frac{\partial \dot{X}_1}{\partial h}, \quad \frac{\partial M_{12}}{\partial h} = \frac{2}{n^2 a}\frac{\partial \dot{Y}_1}{\partial h}, \quad \frac{\partial M_{13}}{\partial h} = 0$$

$$\frac{\partial M_{21}}{\partial h} = \frac{-hM_{21}}{1-h^2-k^2} + \sqrt{\frac{1-h^2-k^2}{na^2}}\left[\frac{\partial^2 X_1}{\partial h \partial k} + \frac{\partial \dot{X}_1}{\partial h}\frac{(\sin F - h\beta)}{n} - \frac{\dot{X}_1}{n}\frac{(\beta + h^2\beta^3)}{1-\beta}\right]$$

$$\frac{\partial M_{22}}{\partial h} = \frac{-hM_{22}}{1-h^2-k^2} + \sqrt{\frac{1-h^2-k^2}{na^2}}\left[\frac{\partial^2 Y_1}{\partial h \partial k} + \frac{\partial \dot{Y}_1}{\partial h}\frac{(\sin F - h\beta)}{n} - \frac{\dot{Y}_1}{n}\frac{(\beta + h^2\beta^3)}{1-\beta}\right]$$

$$\frac{\partial M_{23}}{\partial h} = \frac{hM_{23}}{1-h^2-k^2} + \frac{k}{na^2\sqrt{1-h^2-k^2}}\left(q\frac{\partial Y_1}{\partial h} - p\frac{\partial X_1}{\partial h}\right)$$

$$\frac{\partial M_{31}}{\partial h} = \frac{-h}{1-h^2-k^2}M_{31} - \sqrt{\frac{1-h^2-k^2}{na^2}}\left[\frac{\partial^2 X_1}{\partial h^2} - \frac{\partial \dot{X}_1}{\partial h}\frac{(\cos F - k\beta)}{n} + \frac{\dot{X}_1}{n}\frac{hk\beta^3}{1-\beta}\right]$$

$$\frac{\partial M_{32}}{\partial h} = \frac{-h}{1-h^2-k^2}M_{32} - \sqrt{\frac{1-h^2-k^2}{na^2}}\left[\frac{\partial^2 Y_1}{\partial h^2} - \frac{\partial \dot{Y}_1}{\partial h}\frac{(\cos F - k\beta)}{n} + \frac{\dot{Y}_1}{n}\frac{hk\beta^3}{1-\beta}\right]$$

$$\frac{\partial M_{33}}{\partial h} = \frac{h}{1-h^2-k^2}M_{33} + \frac{M_{33}}{h} - \frac{h}{na^2\sqrt{1-h^2-k^2}}\left(q\frac{\partial Y_1}{\partial h} - p\frac{\partial X_1}{\partial h}\right)$$

$$\frac{\partial M_{41}}{\partial h} = 0, \quad \frac{\partial M_{42}}{\partial h} = 0, \quad \frac{\partial M_{43}}{\partial h} = \frac{hM_{43}}{1-h^2-k^2} + \frac{M_{43}}{Y_1}\frac{\partial Y_1}{\partial h}$$

$$\frac{\partial M_{51}}{\partial h} = 0, \quad \frac{\partial M_{52}}{\partial h} = 0, \quad \frac{\partial M_{53}}{\partial h} = \frac{hM_{53}}{1-h^2-k^2} + \frac{M_{53}}{X_1}\frac{\partial X_1}{\partial h}$$

Table 5. Partial of M with Respect to k

$$\frac{\partial M_{11}}{\partial k} = \frac{2}{n^2 a}\frac{\partial \dot{X}_1}{\partial k}, \quad \frac{\partial M_{12}}{\partial k} = \frac{2}{n^2 a}\frac{\partial \dot{Y}_1}{\partial k}, \quad \frac{\partial M_{13}}{\partial k} = 0$$

$$\frac{\partial M_{21}}{\partial k} = \frac{-kM_{21}}{1-h^2-k^2} + \sqrt{\frac{1-h^2-k^2}{na^2}}\left[\frac{\partial^2 X_1}{\partial k^2} + \frac{\partial \dot{X}_1}{\partial k}\frac{(\sin F - h\beta)}{n} - \frac{\dot{X}_1}{n}\frac{hk\beta^3}{1-\beta}\right]$$

$$\frac{\partial M_{22}}{\partial k} = \frac{-kM_{22}}{1-h^2-k^2} + \sqrt{\frac{1-h^2-k^2}{na^2}}\left[\frac{\partial^2 Y_1}{\partial k^2} + \frac{\partial \dot{Y}_1}{\partial k}\frac{(\sin F - h\beta)}{n} - \frac{\dot{Y}_1}{n}\frac{hk\beta^3}{1-\beta}\right]$$

$$\frac{\partial M_{23}}{\partial k} = \frac{M_{23}}{k} + \frac{kM_{23}}{1-h^2-k^2} + \frac{k}{na^2\sqrt{1-h^2-k^2}}\left(q\frac{\partial Y_1}{\partial k} - p\frac{\partial X_1}{\partial k}\right)$$

$$\frac{\partial M_{31}}{\partial k} = -\frac{kM_{31}}{1-h^2-k^2} - \sqrt{\frac{1-h^2-k^2}{na^2}}\left[\frac{\partial^2 X_1}{\partial k \partial h} - \frac{\partial \dot{X}_1}{\partial k}\left(\frac{\cos F - k\beta}{n}\right) + \frac{\dot{X}_1}{n}\frac{(\beta + k^2\beta^3)}{1-\beta}\right]$$

$$\frac{\partial M_{32}}{\partial k} = \frac{-kM_{32}}{1-h^2-k^2} - \sqrt{\frac{1-h^2-k^2}{na^2}}\left[\frac{\partial^2 Y_1}{\partial k \partial h} - \frac{\partial \dot{Y}_1}{\partial k}\left(\frac{\cos F - k\beta}{n}\right) + \frac{\dot{Y}_1}{n}\frac{(\beta + k^2\beta^3)}{1-\beta}\right]$$

$$\frac{\partial M_{33}}{\partial k} = \frac{kM_{33}}{1-h^2-k^2} - \frac{h}{na^2\sqrt{1-h^2-k^2}}\left(q\frac{\partial Y_1}{\partial k} - p\frac{\partial X_1}{\partial k}\right)$$

$$\frac{\partial M_{41}}{\partial k} = 0, \quad \frac{\partial M_{42}}{\partial k} = 0, \quad \frac{\partial M_{43}}{\partial k} = \frac{kM_{43}}{1-h^2-k^2} + \frac{M_{43}}{Y_1}\frac{\partial Y_1}{\partial k}$$

$$\frac{\partial M_{51}}{\partial k} = 0, \quad \frac{\partial M_{52}}{\partial k} = 0, \quad \frac{\partial M_{53}}{\partial k} = \frac{kM_{53}}{1-h^2-k^2} + \frac{M_{53}}{X_1}\frac{\partial X_1}{\partial k}$$

Table 6. Non-zero Partials of M with Respect to p

$$\frac{\partial M_{23}}{\partial p} = \frac{-kX_1}{\sqrt{1-h^2-k^2}\ na^2}$$

$$\frac{\partial M_{33}}{\partial p} = \frac{hX_1}{\sqrt{1-h^2-k^2}\ na^2}$$

$$\frac{\partial M_{43}}{\partial p} = \frac{pY_1}{\sqrt{1-h^2-k^2}\ na^2}$$

$$\frac{\partial M_{53}}{\partial p} = \frac{pX_1}{\sqrt{1-h^2-k^2}\ na^2}$$

Table 7. Non-zero Partials of M with Respect to q

$$\frac{\partial M_{23}}{\partial q} = \frac{kY_1}{\sqrt{1-h^2-k^2}\ na^2}$$

$$\frac{\partial M_{33}}{\partial q} = \frac{-hY_1}{\sqrt{1-h^2-k^2}\ na^2}$$

$$\frac{\partial M_{43}}{\partial q} = \frac{qY_1}{\sqrt{1-h^2-k^2}\ na^2}$$

$$\frac{\partial M_{53}}{\partial q} = \frac{qX_1}{\sqrt{1-h^2-k^2}\ na^2}$$

Table 8. Partials of $\dot{X}_1$ and $\dot{Y}_1$ with Respect to h and k

$$\frac{\partial \dot{X}_1}{\partial h} = \frac{a}{r}\dot{X}_1 \sin F + \frac{na^2}{r}\left[(h\sin F + k\cos F)\left(\beta + \frac{h^2\beta^3}{1-\beta}\right) + h\beta\sin F\right]$$

$$\frac{\partial \dot{Y}_1}{\partial h} = \frac{a}{r}\dot{Y}_1 \sin F + \frac{na^2}{r}\left[-k\beta\sin F - \frac{hk\beta^3}{1-\beta}(h\sin F + k\cos F)\right]$$

$$\frac{\partial \dot{X}_1}{\partial k} = \frac{a}{r}\dot{X}_1 \cos F + \frac{na^2}{r}\left[h\beta\cos F + \frac{hk\beta^3}{1-\beta}(h\sin F + k\cos F)\right]$$

$$\frac{\partial \dot{Y}_1}{\partial k} = \frac{a}{r}\dot{Y}_1 \cos F + \frac{na^2}{r}\left[-(h\sin F + k\cos F)\left(\beta + \frac{k^2\beta^3}{1-\beta}\right) - k\beta\cos F\right]$$

Table 9. Second Partials of X_1 and Y_1 with Respect to h and k

$$\frac{\partial^2 X_1}{\partial h^2} = a\left[\frac{h\beta^3}{1-\beta}\ (-h\cos F + k\sin F)\left(3 + h^2\beta^2\frac{(3-2\beta)}{(1-\beta)^2}\right) - 2\cos F\ \frac{(\beta+h^2\beta^3)}{1-\beta}\right]$$

$$\frac{\partial^2 Y_1}{\partial h^2} = a\left[\frac{k\beta^3}{1-\beta}\ (h\cos F - k\sin F)\left(1 + h^2\frac{(3\beta^2-2\beta^3)}{(1-\beta)^2}\right) + \frac{2hk\beta^3}{1-\beta}\ \cos F\right]$$

$$\frac{\partial^2 X_1}{\partial k^2} = a\left[\frac{h\beta^3}{1-\beta}\ (-h\cos F + k\sin F)\left(1 + k^2\frac{(3\beta^2-2\beta^3)}{(1-\beta)^2}\right) + \frac{2hk\beta^3}{1-\beta}\ \sin F\right]$$

$$\frac{\partial^2 Y_1}{\partial k^2} = a\left[\frac{k\beta^3}{1-\beta}\ (h\cos F - k\sin F)\left(3 + k^2\frac{(3\beta^2-2\beta^3)}{(1-\beta)^2}\right) - 2\sin F\ \frac{(\beta+k^2\beta^3)}{1-\beta}\right]$$

$$\frac{\partial^2 X_1}{\partial h\,\partial k} = \frac{\partial^2 X_1}{\partial k\,\partial h} = a\left[\frac{k\beta^3}{1-\beta}\ (-h\cos F + k\sin F)\left(1 + h^2\frac{(3\beta^2-2\beta^3)}{(1-\beta)^2}\right) - \frac{hk\beta^3}{1-\beta}\ \cos F + \sin F\ \frac{(\beta+h^2\beta^3)}{1-\beta}\right]$$

$$\frac{\partial^2 Y_1}{\partial k\,\partial h} = \frac{\partial^2 Y_1}{\partial h\,\partial k} = a\left[\frac{h\beta^3}{1-\beta}\ (h\cos F - k\sin F)\left(1 + k^2\frac{(3\beta^2-2\beta^3)}{(1-\beta)^2}\right) - \frac{hk\beta^3}{1-\beta}\ \sin F + \cos F\ \frac{(\beta+k^2\beta^3)}{1-\beta}\right]$$

Table 10. The Shadow Quartic Equation

$$b_1 = 1-h^2\beta$$
$$b_2 = hk\beta$$
$$b_3 = 1-k^2\beta$$
$$d_1 = 1-X_s^2$$
$$d_2 = 1-Y_s^2$$
$$d_3 = 2Y_sX_s$$
$$h_1 = d_1(b_1^2-b_2^2) + d_2(b_2^2-b_3^2) - d_3(b_1b_2-b_2b_3)$$
$$h_2 = -2d_1kb_1 - 2d_2hb_2 + d_3(kb_2+hb_1)$$
$$h_3 = d_1(b_2+k^2) + d_2(b_3^2+h^2) - d_3(b_2b_3+hk) - \frac{a_e^2}{a^2}$$
$$h_4 = 2b_1b_2d_1 + 2b_2b_3 - d_3(b_2^2+b_1b_3)$$
$$h_5 = -2kb_2d_1 - 2hb_3d_2 + d_3(kb_3+hb_2)$$
$$A_0 = h_1^2 + h_4^2$$
$$A_1 = 2h_1h_2 + 2h_4h_5$$
$$A_2 = h_2^2 + 2h_3h_1 - h_4^2 + h_5^2$$
$$A_3 = 2h_3h_2 - 2h_4h_5$$
$$A_4 = h_3^2 - h_5^2$$

$$S^* \equiv A_0\cos^4 F + A_1\cos^3 F + A_2\cos^2 F + A_3\cos F + A_4 = 0$$

Table 11. Partials of the Shadow Function

$$\frac{\partial S}{\partial F} = 2\left[(1-X_s^2)\,X_1 - X_s Y_s Y_1\right]\frac{\partial X_1}{\partial F} + 2\left[(1-Y_s^2)\,Y_1 - X_s Y_s Y_1\right]\frac{\partial Y_1}{\partial F}$$

$$\frac{\partial X_1}{\partial F} = a\left[-(1-h^2\beta)\sin F + hk\beta\cos F\right]$$

$$\frac{\partial Y_1}{\partial F} = a\left[-hk\beta\sin F + (1-k^2\beta)\cos F\right]$$

$$\frac{\partial S}{\partial a} = \frac{2a_e^2}{a}$$

$$\frac{\partial S}{\partial h} = 2\left[(1-X_s^2)\,X_1 - X_s Y_s Y_1\right]\frac{\partial X_1}{\partial h} + 2\left[(1-Y_s^2)\,Y_1 - X_s Y_s X_1\right]\frac{\partial Y_1}{\partial k}$$

$$\frac{\partial S}{\partial k} = 2\left[(1-X_s^2)\,X_1 - X_s Y_s Y_1\right]\frac{\partial X_1}{\partial k} + 2\left[(1-Y_s^2)\,Y_1 - X_s Y_s Y_1\right]\frac{\partial Y_1}{\partial k}$$

$$\frac{\partial S}{\partial p} = 4\left[X_1^2 X_s + X_1 Y_1 Y_s\right]\left[\frac{qY_s + Z_s}{1+p^2+q^2}\right] - 4\left[Y_1^2 Y_s + X_1 Y_1 Y_s\right]\left[\frac{qX_s}{1+p^2+q^2}\right]$$

$$\frac{\partial S}{\partial q} = -4\left[X_1^2 X_s + X_1 Y_1 Y_s\right]\left[\frac{pY_s}{1+p^2+q^2}\right] - 4\left[Y_1^2 Y_s + X_1 Y_1 X_s\right]\left[\frac{-pX_s + Z_s}{1+p^2+q^2}\right]$$

Table 12. J_2 Variation of Parameters Equations

$$\dot{h}_{J_2} = \frac{3\mu R_e^2 J_2 k \quad 1-6(p^2+q^2) + 3(p^2+q^2)^2}{2na^5(1-h^2-k^2)^2\,(1+p^2+q^2)^2}$$

$$\dot{k}_{J_2} = -\frac{3\mu R_e^2 J_2 h \quad 1-6(p^2+q^2) + 3(p^2+q^2)^2}{2na^5(1-h^2-k^2)^2\,(1+p^2+q^2)^2}$$

$$\dot{p}_{J_2} = -\frac{3\mu R_e^2 J_2\, q(1-p^2-q^2)}{2na^5(1-h^2-k^2)^2\,(1+p^2+q^2)}$$

$$\dot{q}_{J_2} = \frac{3\mu R_e^2 J_2\, p(1-p^2-q^2)}{2na^5(1-h^2-k^2)^2\,(1+p^2+q^2)}$$

Table 13. Partial of J_2 Equations with Respect to a

$$\frac{\partial \dot{h}}{\partial a} = -\frac{7}{2}\frac{\dot{h}}{a}$$

$$\frac{\partial \dot{k}}{\partial a} = -\frac{7}{2}\frac{\dot{k}}{a}$$

$$\frac{\partial \dot{p}}{\partial a} = -\frac{7}{2}\frac{\dot{p}}{a}$$

$$\frac{\partial \dot{q}}{\partial a} = -\frac{7}{2}\frac{\dot{q}}{a}$$

Table 11. Partials of the Shadow Function

$$\frac{\partial S}{\partial F} = 2\left[(1-X_s^2)X_1 - X_sY_sY_1\right]\frac{\partial X_1}{\partial F} + 2\left[(1-Y_s^2)Y_1 - X_sY_sY_1\right]\frac{\partial Y_1}{\partial F}$$

$$\frac{\partial X_1}{\partial F} = a\left[-(1-h^2\beta)\sin F + hk\beta\cos F\right]$$

$$\frac{\partial Y_1}{\partial F} = a\left[-hk\beta\sin F + (1-k^2\beta)\cos F\right]$$

$$\frac{\partial S}{\partial a} = \frac{2a_e^2}{a}$$

$$\frac{\partial S}{\partial h} = 2\left[(1-X_s^2)X_1 - X_sY_sY_1\right]\frac{\partial X_1}{\partial h} + 2\left[(1-Y_s^2)Y_1 - X_sY_sX_1\right]\frac{\partial Y_1}{\partial k}$$

$$\frac{\partial S}{\partial k} = 2\left[(1-X_s^2)X_1 - X_sY_sY_1\right]\frac{\partial X_1}{\partial k} + 2\left[(1-Y_s^2)Y_1 - X_sY_sY_1\right]\frac{\partial Y_1}{\partial k}$$

$$\frac{\partial S}{\partial p} = 4\left[X_1^2X_s + X_1Y_1Y_s\right]\left[\frac{qY_s + Z_s}{1+p^2+q^2}\right] - 4\left[Y_1^2Y_s + X_1Y_1Y_s\right]\left[\frac{qX_s}{1+p^2+q^2}\right]$$

$$\frac{\partial S}{\partial q} = -4\left[X_1^2X_s + X_1Y_1Y_s\right]\left[\frac{pY_s}{1+p^2+q^2}\right] - 4\left[Y_1^2Y_s + X_1Y_1X_s\right]\left[\frac{-pX_s+Z_s}{1+p^2+q^2}\right]$$

Table 12. J_2 Variation of Parameters Equations

$$\dot{h}_{J_2} = \frac{3\mu R_e^2 J_2 k \;\; 1-6(p^2+q^2) + 3(p^2+q^2)^2}{2na^5(1-h^2-k^2)^2\,(1+p^2+q^2)^2}$$

$$\dot{k}_{J_2} = -\frac{3\mu R_e^2 J_2 h \;\; 1-6(p^2+q^2) + 3(p^2+q^2)^2}{2na^5(1-h^2-k^2)^2\,(1+p^2+q^2)^2}$$

$$\dot{p}_{J_2} = -\frac{3\mu R_e^2 J_2\, q(1-p^2-q^2)}{2na^5(1-h^2-k^2)^2\,(1+p^2+q^2)}$$

$$\dot{q}_{J_2} = \frac{3\mu R_e^2 J_2\, p(1-p^2-q^2)}{2na^5(1-h^2-k^2)^2\,(1+p^2+q^2)}$$

Table 13. Partial of J_2 Equations with Respect to a

$$\frac{\partial \dot{h}}{\partial a} = -\frac{7}{2}\frac{\dot{h}}{a}$$

$$\frac{\partial \dot{k}}{\partial a} = -\frac{7}{2}\frac{\dot{k}}{a}$$

$$\frac{\partial \dot{p}}{\partial a} = -\frac{7}{2}\frac{\dot{p}}{a}$$

$$\frac{\partial \dot{q}}{\partial a} = -\frac{7}{2}\frac{\dot{q}}{a}$$

Table 14. Partial of J_2 Equations with Respect to h

$$\frac{\partial \dot{h}}{\partial h} = \frac{4h\dot{h}}{1-h^2-k^2}$$

$$\frac{\partial \dot{k}}{\partial h} = \frac{\dot{k}}{h} + \frac{4h\dot{k}}{1-h^2-k^2}$$

$$\frac{\partial \dot{p}}{\partial h} = \frac{4h\dot{p}}{1-h^2-k^2}$$

$$\frac{\partial \dot{q}}{\partial h} = \frac{4h\dot{q}}{1-h^2-k^2}$$

Table 15. Partial of J_2 Equations with Respect to k

$$\frac{\partial \dot{h}}{\partial k} = \frac{\dot{h}}{k} + \frac{4k\dot{h}}{1-h^2-k^2}$$

$$\frac{\partial \dot{k}}{\partial k} = \frac{4k\dot{k}}{1-h^2-k^2}$$

$$\frac{\partial \dot{p}}{\partial k} = \frac{4k\dot{p}}{1-h^2-k^2}$$

$$\frac{\partial \dot{q}}{\partial k} = \frac{4k\dot{q}}{1-h^2-k^2}$$

Table 16. Partial of J_2 Equations with Respect to p

$$\frac{\partial \dot{h}}{\partial p} = \left(\frac{12\mu R_e^2 J_2}{na^5}\right) \frac{kp(3(p^2+q^2)-2)}{(1-h^2-k^2)^2\ (1+p^2+q^2)^3}$$

$$\frac{\partial \dot{k}}{\partial p} = \left(\frac{12\mu R_e^2 J_2}{na^5}\right) \frac{hp(3(p^2+q^2)-2)}{(1-h^2-k^2)^2\ (1+p^2+q^2)^3}$$

$$\frac{\partial \dot{p}}{\partial p} = \frac{-2\dot{p}}{1-(p^2+q^2)^2}$$

$$\frac{\partial \dot{q}}{\partial p} = \frac{-\dot{q}}{p}\left[\frac{1+(p^2+q^2)^2}{1-(p^2+q^2)^2}\right]$$

Table 17. Partial of J_2 Equations with Respect to q

$$\frac{\partial \dot{h}}{\partial q} = \left(\frac{12\mu R_e^2 J_2}{na^5}\right) \frac{kq(3(p^2+q^2)-2)}{(1-h^2-k^2)^2\ (1+p^2+q^2)^3}$$

$$\frac{\partial \dot{k}}{\partial q} = \left(\frac{12\mu R_e^2 J_2}{na^5}\right) \frac{hq(3(p^2+q^2)-2)}{(1-h^2-k^2)^2\ (1+p^2+q^2)^3}$$

$$\frac{\partial \dot{p}}{\partial q} = \frac{-\dot{p}}{q}\left[\frac{1+(p^2+q^2)^2}{1-(p^2+q^2)^2}\right]$$

$$\frac{\partial \dot{q}}{\partial q} = \frac{-2\dot{q}}{1-(p^2+q^2)^2}$$

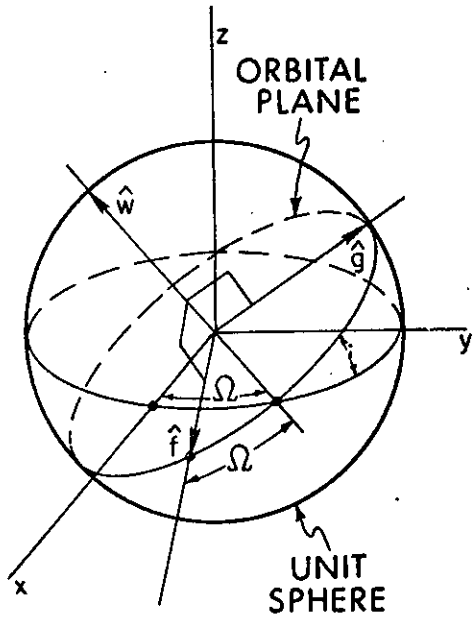

Figure 1. Equinoctial Coordinate Frame

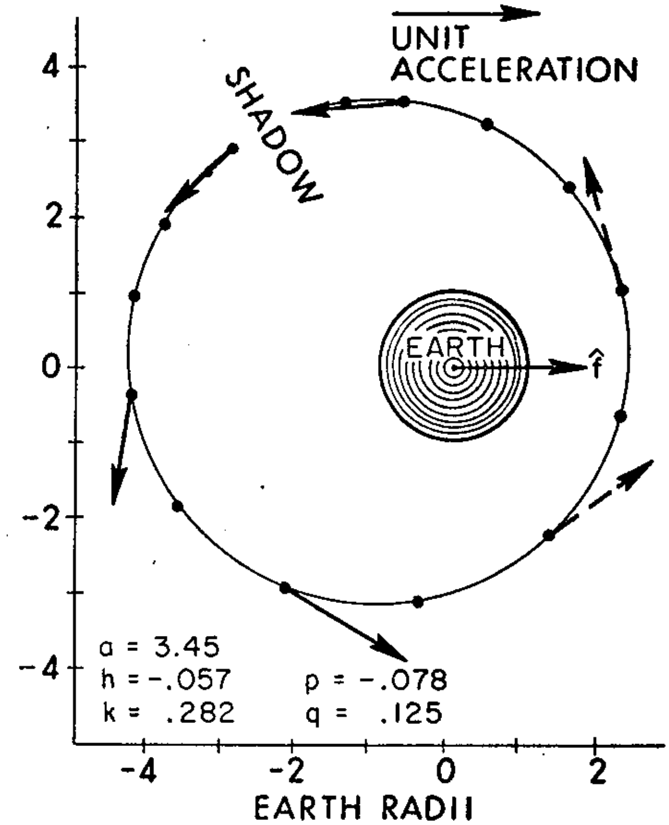

Figure 3. Orbital Plane Projection of Unit Thrust Acceleration Vector for Orbit at t = 31.7 Days

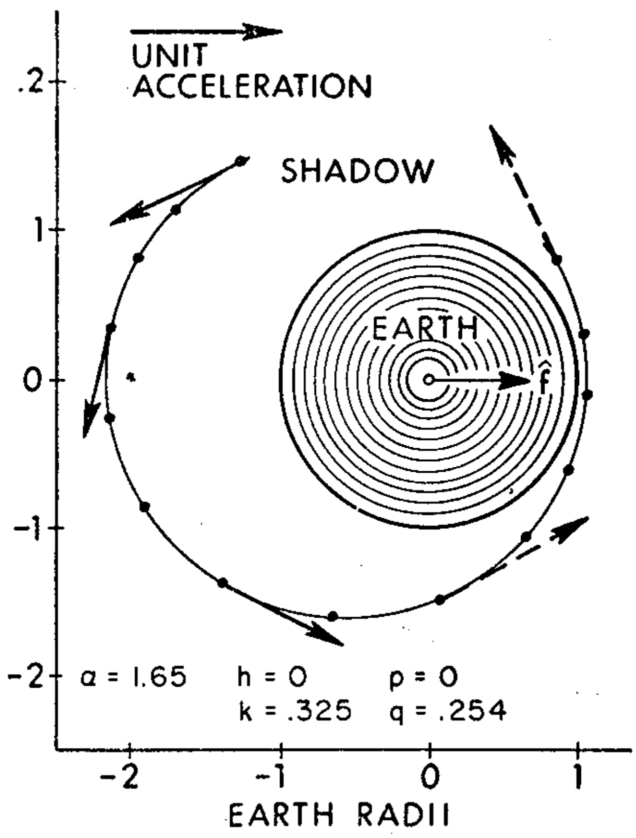

Figure 2. Orbital Plane Projection of Unit Thrust Acceleration Vector for Initial Orbit

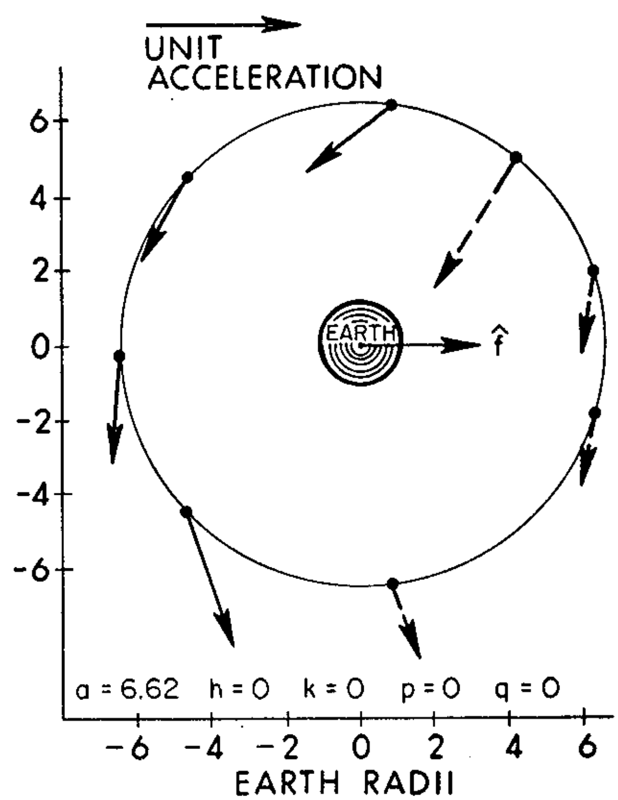

Figure 4. Orbital Plane Projection of Unit Thrust Acceleration Vector for Final Orbit

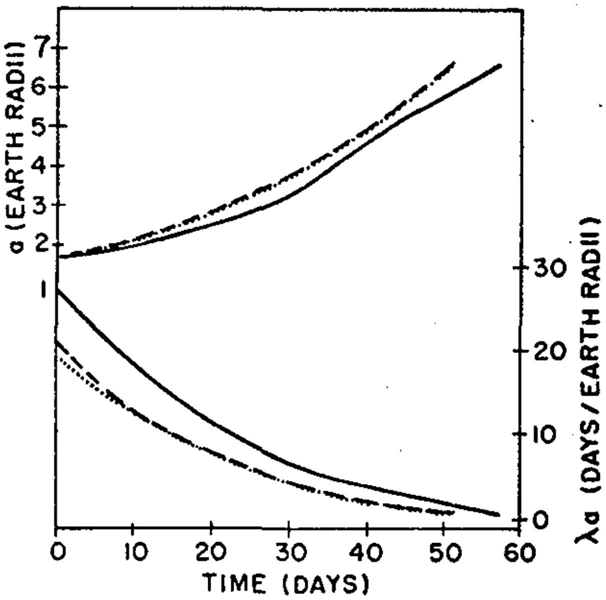

Figure 5. a' and λ_a versus Time*

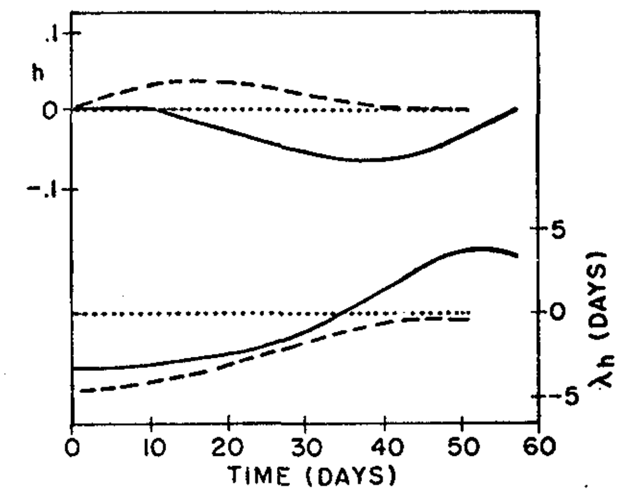

Figure 6. h and λ_h versus Time*

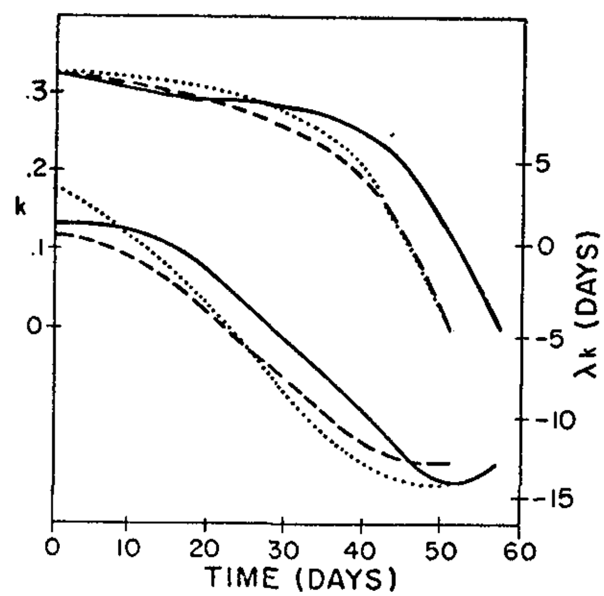

Figure 7. k and λ_k versus Time*

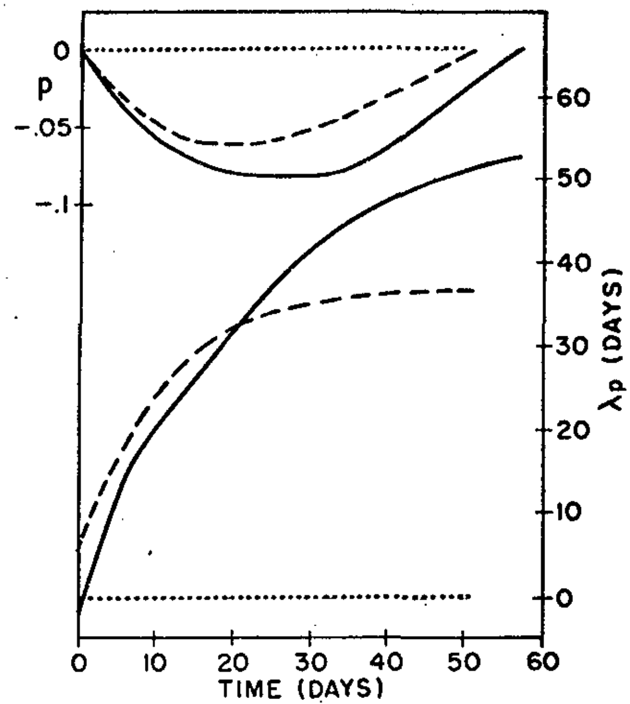

Figure 8. p and λ_p versus Time*

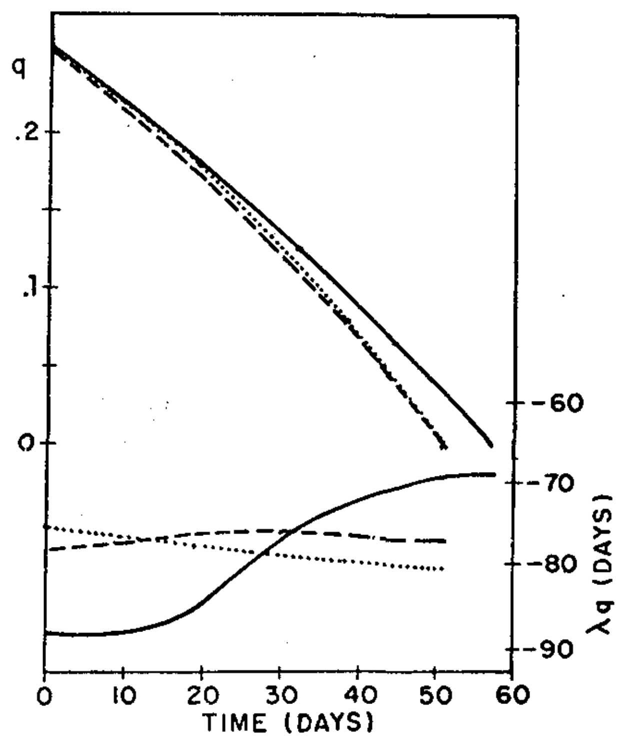

Figure 9. q and λ_q versus Time*

*The upper set of three curves in the figure is the state, the lower set of three curves is the costate. Dotted line is case with no oblateness or shadowing. Dashed line is case with oblateness and solid line is case with oblateness and shadowing.

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