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APPLICATION OF THE SEMIANALYTICAL SATELLITE THEORY TO SHALLOW RESONANCE ORBITS

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Application of the Semianalytical Satellite Theory to Shallow Resonance Orbits

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Abstract

This paper discusses the accurate and efficient modelling of the tesseral linear combination short periodics due to high order shallow resonant terms. Previous development of the Semianalytical Satellite Theory (SST) split the dynamical perturbations into the Mean Element Equations of Motion and the Short Periodic Motion. The smooth force models for the mean elements are integrated with very large step sizes (1 day in length), while the short periodics are recovered analytically. Each portion of the theory employs recursions on the special functions to allow for a high degree of model selectivity. Shallow resonant orbits pose unique challenges for the SST because:

1. The primary shallow resonance frequency may constrain the mean element numerical integration size
2. The tesseral harmonics with order adjacent to the resonant order are likely to contribute significant short period motion

Because the current SST tesseral harmonic linear combination term short periodic model software has only the maximum degree and order as input parameters, all the intermediate short period terms must be included. Complete modelling of the shallow resonant terms may exceed the software limitations of the current SST implementation (400 linear combination terms). To study the performance of the SST for shallow resonant cases, a modified form of the SST tesseral linear combination short periodic software has been constructed in which the number of allowed linear combination terms is increased to 1600. For a 21 x 21 geopotential field case, inclusion of the shallow resonance terms in the short periodic model suggests the existence of significant J2 secular/shallow resonance coupling terms. A compact analytical model is developed for these terms.

INTRODUCTION

The development of Semianalytical Satellite Theory (SST) is motivated by the desire to develop a perturbation theory capable of demonstrating accuracy across a broad range of satellite orbits. Previous development of the Semianalytical Satellite Theory (SST) split the dynamical perturbations into the Mean Element Equations of Motion and the Short Periodic Motion. The smooth force models for the mean elements are integrated with very large step sizes (1 day in length), while the short periodics are recovered analytically. The application of a semianalytical satellite theory to high accuracy orbit prediction and determination problems requires extensive dynamical modelling for the short-period motion as well as the long-period and secular motion. In application, the selected models are orbit dependent. In this paper, we will focus on the tesseral harmonic perturbations, and their application to satellites in shallow resonance.

Tesseral Harmonics Model

The disturbing potential U due to the gravitational field can be expressed by

$$U = \frac{\mu}{r}\sum_{l=2}^{N}\sum_{m=0}^{l}(\frac{R_e}{r})^l P_{l,m}(\sin\phi)[C_{l,m}\cos(m\alpha) + S_{l,m}\sin(m\alpha)] \quad (1)$$

where

μ = Earth's gravitational constant
r = distance from Earth to satellite
R_e = equatorial radius of the Earth
ϕ = geocentric latitude of the satellite
α = geographic longitude of the satellite
$P_{l,m}$ = associated Legendre function of degree l and order m
$C_{l,m}$, $S_{l,m}$ = empirically determined gravity harmonic coefficients

The $(C,S)_{l,m}$ terms for which $m \neq 0$ are called the tesseral harmonics and represent the longitude dependent deviations from sphericity. The tesseral harmonics generally result in short-periodic perturbations of the orbit. However, when the satellite's mean motion is nearly a multiple of the Earth's rotation rate, the tesseral perturbations can have much larger periods and magnitudes. This phenomena is called tesseral resonance.

A general formulation of the tesseral disturbing potential in terms of special functions of the equinoctial elements was given by Cefola [3]. It was elaborated in McClain [11], and [12], leading to an initial implementation by Dunham of the resonance potential [13],[10]. That capability was limited; only 10 resonant pairs of coefficients were included, each pair being separately implemented; and the model was valid only for eccentricity $\leq .5$. The tesseral resonance model subsequently was improved by Proulx [16], whose work included application of a recursion formula for the Hansen coefficients [8]. Proulx and McClain [17], [18] also developed a modified Hansen coefficient expansion with improved eccentricity convergence. With these developments, full field implementation of the tesseral resonance model was achieved, valid for all orbits with eccentricity < 1.0

The disturbing potential due to the Earth's gravitational field may be cast into a Fourier series

$$U = \sum_{j=-\infty}^{\infty} \sum_{m=0}^{\infty} U_{m,t} \tag{2}$$

where

$$U_{m,t} = V_{m,j} \cos(\psi_{j,m}) + W_{m,j} \sin(\psi_{j,m}) \tag{3}$$

with phase angles $\psi_{t,m} = j\lambda - m\theta$ given in terms of λ, the mean longitude, and θ, the Greenwich hour angle.

Based on the commensurability between the satellite's mean motion and the Earth's rotation rate, the phase angles appearing in the potential can be expressed as the disjoint union

$$\{\psi_{j,m}\} = \{\mu_{j,m}\} \cup \{\phi_{j,m}\} \tag{4}$$

of the slowly varying (resonant) phase angles, $\mu_{j,m}$, and the rapidly varying (short-periodic) phase angles, $\phi_{j,m}$.

In reference [19] the mean element equations of motion to first order in the geopotential harmonic coefficients were given by

$$\dot{\bar{a}}_i = \bar{n}\delta_{i,6} - \sum_{k=1}^{6} (\bar{a}_i, \bar{a}_k) \frac{\partial}{\partial \bar{a}_k} [U_{0,0}(\vec{\bar{a}}) + \sum_{j,m} U_{m,j}(\vec{\bar{a}}, \bar{\mu}_{j,m})] \tag{5}$$

where $(\bar{a}_i, \bar{a}_k)$ are the Poisson Brackets in the equinoctial elements derived by Broucke and Cefola [2]; and $\vec{\bar{a}}$ is the vector the mean equinoctial elements. The term $\bar{n}\delta_{i,6}$ is due to two-body mechanics, while $U_{0,0}(\vec{\bar{a}})$ is the averaged potential due to the zonal harmonics in the geopotential. The first-order resonance contribution to the mean element rates due to the slowly varying phase angle $\mu_{j,m}$ is given by

$$\dot{\bar{a}}_i = - \sum_{k=1}^{6} (\bar{a}_i, \bar{a}_k) \frac{\partial}{\partial \bar{a}_k} U_{m,j}(\vec{\bar{a}}, \bar{\mu}_{j,m}) \tag{6}$$

The short-periodic motion is recovered analytically as a Fourier series in the rapidly varying phase angles $\phi_{j,m}$

$$\Delta a_i = \sum_{j,m} C^i_{j,m} \cos(\phi_{j,m}) + D^i_{j,m} \sin(\phi_{j,m}) \tag{7}$$

The coefficients of this series are functions of the five slowly varying mean equinoctial elements. For $m = 0$, Slutsky [14] developed a closed form expression in the eccentricity for the zonal short periodic coefficients, in terms of the true longitude. Proulx et. al. developed recursive analytic expressions for the tesseral linear combination terms [19]. For $j = 0$,the m-daily expressions are closed form in the eccentricity; the remaining tesseral short periodic variations are developed in power series in the eccentricity.

The SST tesseral perturbation models have been tested exhaustively for low altitude, non-resonant cases [19] and for high altitude deep resonance cases, e.g.Geosynchronous, Molniya (see [5],[6]). There has recently been interest in satellites whose orbits have long repeat ground tracks. Such orbits tend to be in shallow resonance with the Earth. Recent analyses of the tesseral geopotential terms such as the 1987 ERS-1 study by Wakker [20] have clearly shown the importance of

- Low degree and order geopotential terms
- High degree and order terms centered around the shallow resonant order

The currently available algorithmic approach to tesseral perturbations was designed to include low degree and order terms in the short periodic model, and to include the resonant frequency in the mean element equations of motion (see the left side of Figure 1). For the typical non-resonant, or deeply resonant satellites, this presents no difficulty. However, to model high order shallow resonant terms, together with their side bands, requires the computation of many negligible intermediate terms in order to include the high degree and order terms centered around the resonant order (see the right side of Figure 1). The software implementation of the SST model limits the total number of linear combination short periodic frequencies to 400 terms. Inclusion of the full geopotential model to take in the tesseral perturbations around a shallow resonance breaks that limit. Possible solutions to this problem have been considered. One solution forces shallow resonance terms into the resonance model of the mean equations of motion. This has the effect of including frequencies as short as once per day in these equations, thus limiting the numerical integration stepsize which can be applied. Alternatively, the total number of tesseral short periodic frequencies may be increased to include the full model. This causes the computation of many negligible terms. The best solution is the development of an architecture which will select the important low degree and order tesseral short periodics, together with the non-negligible band of harmonics about the shallow resonance. To facilitate the results in this paper, the second alternative was chosen, increasing the allowed tesseral harmonic frequencies to 1600 terms. This allows implementation of a full 21 x 21 tesseral model.

Numerical Results

In this section we show the results of an investigation into the accuracy of the SST when applied to an orbit in shallow resonance. We chose for investigation

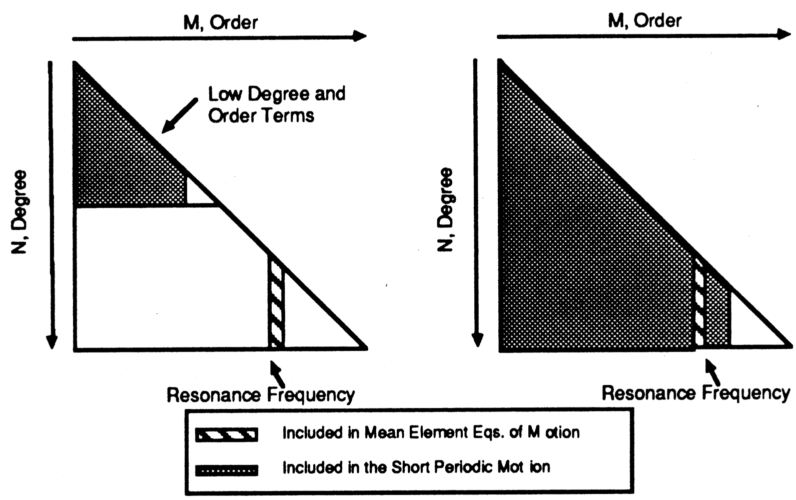

Figure 1: Tesseral Software Architecture

a 13.4 rev/day, critical inclination orbit derived from King-Hele [9], and Oesterwinter [15]. The geometry of the orbit is shown in Table 1. The methodology of our investigation consists of the following steps:

- Develop a numerically integrated Cowell trajectory as truth data
- Least Squares fit the SST theory to the truth data
- Examine the difference between the converged, best fit SST trajectory and the truth

The objective of this method is to detect the character of the residual pattern, and the occurence of Gibb's phenomena in the observation span; and to detect the occurence of secular growth due to unmodelled effects in the predict span. As seen in Table 1 full force models were included in the truth trajectory; three test cases were constructed by varying the geopotential field. In each testcase, a five day truth model was created. The first three days were fit with the SST using the force models shown in Table 2. The best estimated SST state was predicted ahead for two days. Summaries of the fit and predict statistics are shown in Table 3, Table 4, and Table 5.

The baseline 8 x 8 test case resulted, as expected, with very good fit and predict residual statistics. Note in particular that the total position rms in the three day observation span is only 0.99 meters, while the predict error rms hovers just above 1 meter. Note that this test demonstrates the ability of the SST to include J_2 terms.

The error statistics for the 21 x 21 case are approximately four times larger than the 8 x 8 baseline test case. Of greater interest is that the dominant error in the along track direction has a period equal to the dominant sideband frequency about the shallow resonant term (approximately 35 hours). We have hypothesized that this error is us due to unmodelled secular J_2 / shallow resonance coupling. The last (small J_2) test case was constructed to test this hypothesis. The idea is that if the large, structured residual error is due to the coupling of the J_2 averaged rates with the tesseral linear combination term, then making J_2 effectively zero should null out that dominant error signature. In this case, note that full 21 x 21 short-periodic model is used. This is done to reduce the noise in the observed error signal. The test supported our hypothesis. The error statistics dropped back down to the near 1 meter values seen in the 8 x 8 baseline test case.

J_2 SECULAR/SHALLOW RESONANCE COUPLING MODEL

Berger[1] and Gooding[7] observed that the J_2 / tesseral coupling causes 12 hour effects in the along-track direction of the satellite. In early testing of the

- Orbit (from RAE Table of Satellites and NSWC/DL-3724)
  - a = 7485 km (alt = 1107 km)
  - e = 0.008
  - i = 63.49 degrees
  - w = 163 degrees
- Cowell Precision Integration (12th order Predictor/Corrector with 60 sec step)
  - Geopotential Choices (all use GEM-10B coefficients)
    1. 8 x 8 geopotential
    2. 21 x 21 geopotential
    3. 21 x 21 geopotential with 'small' $J_2 = o(10^{-5})$
- Truth Trajectory
  - Lunar-Solar Point Masses
  - Mean of 1950 (M50) integration coordinate system
  - M50 position/velocity data at 450 sec intervals over 5 days

Table 1: Shallow Resonance Test Case

	Mean Equations of Motion Force Model		
	8 x 8 Case	21 x 21 Case ($J_2 \neq 0$)	21 x 21 Case $J_2 = o(10^{-5})$
Zonals	$J_2, \ldots, J_8$	$J_2, \ldots, J_{21}$	$J_2, \ldots, J_{21}$
Tesseral Resonance	None	$(13,13), \ldots, (21,13)$	$(13,13), \ldots, (21,13)$
$J2^2$	yes	yes	yes
Third Body	Point mass	Point mass	Point mass

	Short Periodic Force Model		
	8 x 8 Case	21 x 21 Case ($J_2 \neq 0$)	21 x 21 Case $J_2 = o(10^{-5})$
Zonals	$J_2, \ldots, J_8$	$J_2, \ldots, J_{21}$	$J_2, \ldots, J_{21}$
Tesseral m-dailys	8 x 8 field	21 x 21 field	21 x 21 field
Tesseral Harmonics	8 x 8 field	21 x 15 field	21 x 20 field
$J2^2$	yes	yes	yes
Third Body	Point mass	Point mass	Point mass

Table 2: Summary of SST Test Case Force Models

Position Differences (meters)				
Position Component	Observe Span		Predict Span	
	RMS	MAX	RMS	MAX
Radial	0.34	0.94	0.54	1.29
Cross Track	0.22	0.73	0.38	0.81
Along Track	0.90	2.06	1.19	2.70
Total	0.99	2.10	1.36	2.85

Velocity Differences (cm/sec)				
Velocity Component	Observe Span		Predict Span	
	RMS	MAX	RMS	MAX
Radial	0.07	0.16	0.08	0.19
Cross Track	0.04	0.10	0.05	0.13
Along Track	0.04	0.10	0.05	0.13
Total	0.09	0.18	0.11	0.22

Table 3: Summary of Results: SST vs 8 x 8 Truth

Position Differences (meters)				
Position Component	Observe Span		Predict Span	
	RMS	MAX	RMS	MAX
Radial	0.53	1.05	0.66	1.43
Cross Track	0.66	1.78	0.69	1.89
Along Track	3.73	8.54	4.29	9.72
Total	3.83	8.55	4.40	9.78

Velocity Differences (cm/sec)				
Velocity Component	Observe Span		Predict Span	
	RMS	MAX	RMS	MAX
Radial	0.35	0.71	0.40	0.81
Cross Track	0.07	0.16	0.08	0.20
Along Track	0.05	0.15	0.06	0.15
Total	0.36	0.71	0.41	0.83

Table 4: Summary of Results: SST vs 21 x 21 Truth

Position Differences (meters)				
Position	Observe Span		Predict Span	
Component	RMS	MAX	RMS	MAX
Radial	0.11	0.25	0.13	0.33
Cross Track	0.44	1.41	0.39	1.11
Along Track	1.29	2.75	1.27	3.23
Total	1.37	2.83	1.34	3.27

Velocity Differences (cm/sec)				
Velocity	Observe Span		Predict Span	
Component	RMS	MAX	RMS	MAX
Radial	0.12	0.26	0.12	0.29
Cross Track	0.05	0.12	0.04	0.11
Along Track	0.01	0.02	0.01	0.03
Total	0.13	0.26	0.13	0.29

Table 5: Summary of Results: SST vs 21 x 21 Truth (Small J_2)

semianalytical tesseral short periodic software implementation, it was observed that the J_2 / m-daily coupling contributed approximately 11 meters (rms) to the total position error for low altitude satellites. Implementation of this coupling term, derived by Cefola [4], reduced the total rms error from 12.6 meters to 1.8 meters[19]. What follows is a derivation of the J_2 / shallow resonance coupling model, based Cefola's derivation of the J_2 / m-daily coupling.

Let the osculating VOP equations of motion assume the form

$$\dot{a}_i = n\delta_{i,6} + \varepsilon F_i(\vec{a}, \lambda) + \nu G_i(\vec{a}, \psi_{j,m}) \tag{8}$$

where $\vec{a}$ is the vector of equinoctial elements, with $a_1, \ldots, a_5$ the slowly varying elements, and with $a_6 = \lambda$ the mean longitude. Here εF_i is the J_2 contribution, and νG_i is the tesseral contribution associated to the harmonic $\psi_{j,m} = j\lambda - m\theta$. Following the generalized method of averaging, we assume the near-identity transformation

$$a_i = \bar{a}_i + \varepsilon\eta_i(\vec{\bar{a}}, \bar{\lambda}) + \nu\rho_i(\vec{\bar{a}}, \bar{\psi}_{j,m}) + \varepsilon\nu\gamma_i(\vec{\bar{a}}, \bar{\psi}_{j,m}) \tag{9}$$

where $\varepsilon\eta_i$ is the J_2 first order short periodic term, $\nu\rho_i$ is the first order tesseral shallow resonance short periodic term, and $\varepsilon\nu\gamma_i$ is the second order tesseral shallow resonance short periodic term.

The averaged equations of motion takes the form

$$\dot{\bar{a}}_i = \bar{n}\delta_{i,6} + \varepsilon A_i(\vec{\bar{a}}) \tag{10}$$

where

$$A_i(\vec{\bar{a}}) = \frac{1}{2\pi}\int_0^{2\pi} F_i(\vec{\bar{a}}, \bar{\lambda}) \; d\bar{\lambda} \tag{11}$$

with the average taking place over the fast variable λ, is the first order J_2 contribution to the mean equations of motion. The standard procedure in the generalized method of averaging is to obtain two expressions for the osculating element rates. These expressions are set equal on an order by order basis. To shorten this development, only terms proportional to $\varepsilon\nu$ will be quoted, since these are the new terms of interst.

Differentiating Eq. 9, and using Eq. 10 to estimate the mean element rates gives the first expression of rate terms proportional to $\varepsilon\nu$

$$(j\bar{n} - m\omega_e)\frac{\partial\gamma_i}{\partial\bar{\psi}_{j,m}} + jA_6\frac{\partial\rho_i}{\partial\bar{\psi}_{j,m}} + \sum_{k=1}^{5}\frac{\partial\rho_i}{\partial a_k}A_k \tag{12}$$

Note that $\bar{\psi}_{j,m} = j\bar{\lambda} - m\theta$ is the shallow resonance phase angle evaluated at the mean mean longtitude. Expanding Eq. 8 in a Taylor series about the mean elements gives an alternative expression of rate terms proportional to $\varepsilon\nu$

$$\sum_{k=1}^{5}[\frac{\partial F_i}{\partial\bar{a}_k}\rho_k + \frac{\partial G_i}{\partial\bar{a}_k}\eta_k] + \frac{\partial F_i}{\partial\bar{\lambda}}\rho_6 + j\frac{\partial G_i}{\partial\bar{\psi}_{j,m}}\eta_6 - \delta_{i,6}\frac{3}{2}\frac{\bar{n}}{\bar{a}}\gamma_1 \tag{13}$$

Equating expression 12 and expression 13, and applying the averaging operator gives the fundamental differential equation for γ_i

$$(j\bar{n} - m\omega_e)\frac{\partial\gamma_i}{\partial\bar{\psi}_{j,m}} = \sum_{k=1}^{5}[\frac{\partial A_i}{\partial\bar{a}_k}\rho_k - \frac{\partial\rho_i}{\partial\bar{a}_k}A_k] - j\frac{\partial\rho_i}{\partial\bar{\psi}_{j,m}}A_6 - \delta_{i,6}\frac{3}{2}\frac{\bar{n}}{\bar{a}}\gamma_1 \tag{14}$$

Finally, Eq. 14 can be integrated analytically if an analytical form for the tesseral short-periodic, ρ_i, is assumed. Such an expression has been developed explicitly in [19]. For the shallow resonance case, that form is given by

$$\nu\rho_i = \sum_{j,m} C^i_{j,m}\cos(\bar{\psi}_{j,m}) + D^i_{j,m}\sin(\bar{\psi}_{j,m}) \tag{15}$$

Substituting Eq. 15 into Eq. 14 and performing the necessary algebra leads to the Fourier series expansion for these coupling terms

$$\varepsilon\nu\gamma_i = \sum_{j,m}\tilde{C}^i_{j,m}\cos(\bar{\psi}_{j,m}) + \tilde{D}^i_{j,m}\sin(\bar{\psi}_{j,m}) \tag{16}$$

where

$$\tilde{C}^i_{j,m} = -\frac{\varepsilon}{\dot{\bar{\psi}}_{j,m}}\left\{\sum_{k=1}^{5}[\frac{\partial A_i}{\partial\bar{a}_k}D^k_{j,m} - \frac{\partial D^i_{j,m}}{\partial\bar{a}_k}A_k] + jA_6C^i_{j,m}\right\} + \frac{\frac{3}{2}\frac{\bar{n}}{\bar{a}}}{\dot{\bar{\psi}}_{j,m}}\delta_{i,6}\tilde{D}^1_{j,m} \tag{17}$$

and where

$$\tilde{D}^i_{j,m} = \frac{\varepsilon}{\dot{\bar{\psi}}_{j,m}}\left\{\sum_{k=1}^{5}[\frac{\partial A_i}{\partial\bar{a}_k}C^k_{j,m} - \frac{\partial C^i_{j,m}}{\partial\bar{a}_k}A_k] - jA_6D^i_{j,m}\right\} - \frac{\frac{3}{2}\frac{\bar{n}}{\bar{a}}}{\dot{\bar{\psi}}_{j,m}}\delta_{i,6}\tilde{C}^1_{j,m} \tag{18}$$

It is noted that Eq. 17 and Eq. 18 reduce to those obtained by Cefola in [4] when $j = 0$ (J_2/m-daily coupling).

CONCLUSIONS AND FUTURE WORK

This artical has demonstrated that the current tesseral linear combination short periodic model demonstrates approximately one (1) meter level accuracy for satellites in non-resonant orbit, and approximately five (5) meter accuracy for satellites in shallow resonant orbits. There appears to be a significant coupling between the secular J_2 rates and the tesseral harmonics associated to shallow resonance. An analytical model was derived for this phenomenon.

It appears that significant short-period and shallow resonant phenomena for satellites in long repeat ground track orbits require a modification in the software architecture which implements these perturbations. The concept is to develop this architecture so that seperate, selectable, regions of the short-periodic tesseral harmonic field are available: one region for low order terms, and another region for shallow resonance side band terms.

In addition, to achieve the one (1) meter level accuracy, J_2/shallow resonance second-order coupling terms must be included in the tesseral short periodic model. Implementation of this model requires the calculation of partial derivatives of the first-order short-period Fourier series coefficients, as seen in Eq. 17, and Eq. 18. Recursive analytic models for the first order terms exist. Future work includes a recursive analytic developement of their partial derivatives.

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References

[1] Berger, X. *Importance of the Coupling Effects Between Earth Potential Harmonics in the Motion of an Artificial Satellite*, Satellite Dynamics, Editor G.E.O. Gicaglia, COSPAR-IAU-IUTAM Symposium, June 19-21, 1974, Sao Paulo, Brazil, Springer-Verlag, Berlin, 1975

[2] Broucke ,R. and Cefola, P., *On the Equinoctial Orbit Elements*, Celestial Mechanics, vol. 5 1972, pp. 303-310

[3] Cefola, P.,*A Recursive Formulation for the Tesseral Disturbing Function in Equinoctial Variables*, AIAA Paper 76-839, Aug. 1976.

[4] Cefola, P., *Second Order Coupling of M-Daily Terms with J_2 Secular Rates*, Draper Lab IOC No. IRD-011-15z-PJC, July, 1981.

[5] Cefola, P., *The Long-term Orbital Motion of the Desynchronized WESTAR II*, AAS/AIAA paper No. 87-446, Aug. 1987.

[6] Fieger, M., Proulx, R., and Cefola, P., *An Evaluation of Semianalytical Satellite Theory Against Long Arcs of Real Data for Highly Eccentric Orbits*, AAS/AIAA Paper No. 87-444, Aug. 1987.

[7] R.H. Gooding, *On the Generation of Satellite Position (and Velocity) by a Mixed Analytical-Numerical Proceedure*,Adv. Space Res., Vol. 1, pp. 83-93, 1981

[8] Hansen, P., *Entwickelung des Products einer Potenz des Radius Vectors mit dem Sinus oder Cosinus eines Vielfachen der Wahren Anomalie inReihen*, Abhandlungen der Mathematisch-Physischen Classer der Königlich Sächsischen Gesellschaft der Wissenschaften, vol. 2, Leipsig: 1855, pp. 181-281; English tran. by J.C. Van der Ha, Mission Analysis, European Space Operations Center, Ropbert Bosch Str. 5, 6100 Darmstadt, FRG, 1978-1979.

[9] King-Hele, D., Pilkington, J., Walker, D., Hiller, H., and Winterbottom, A., *The R.A.E. Table of Earth Satellites 1957-1982*, Wiley-Interscience, Second Edition 1983

[10] Long, A. and Early, L., *System Description and User Guide for the GTDS RD Averaged Orbit Generator*, CSC/SD-78/6020, Nov. 1978.

[11] McClain, W.,*A Recursively Formulated First Order Semianalytic Artificial Satellite Theory Based on the Generalized Method of Averaging*, Volume 1, NASA-CR-156782, N78-28147, Nov. 1977.

[12] McClain, W.,*A Recursively Formulated First Order Semianalytic Artificial Satellite Theory Based on the Generalized Method of Averaging*, Volume 2, NASA-CR-156783, N78-28148, Nov. 1978.

[13] McClain, W., Long, A., and Early, L., *Development and Evaluation of a Hybrid Averaged Orbit Generator*, AIAA Preprint 78-1382, AIAA/AAS Astrodynamics Conference, Palo Alto, California, Aug 1978.

[14] McClain, W. and Slutsky, M., *A Theory for the Short Period Motion of an Artifical Satellite*, AIAA Paper No. AIAA-80-1658, Aug. 1980.

[15] Oesterwinter, C., *Perturbations in Perigee and Eccentricity at the Critical Inclination*, NSWC/DL-TR-3724, December 1978.

[16] Proulx, R. *Mathematical Description of the Tesseral Resonance and Resonant Harmonic Coefficient Solve-For Capabilities*, Draper IOC NSWC-001-15Z-RJP, Apr. 1982

[17] Proulx, R. and McClain, W., *Series Representation and Rational Approximations for Hansen Coefficients*, AIAA Paper 82-0073, Jan. 1982.

[18] Proulx, R. and McClain, W., *Series Representation and Rational Approximations for Hansen Coefficients*, pp. 313 - 325,Journal of Guidance, Control, and Dynamics July 1988.

[19] Proulx,R., McClain,W., Cefola,P. and Early,L., *A Theory for the Short Periodic Motion Due to the Tesseral Harmonic Gravity Field*, Paper No. AAS-81-180, AAS/AIAA Astrodynamics Specialists Conference, Lake Tahoe, Nevada, August 1981

[20] Wakker,K., Ambrosius, B., Zandbergen, R., and Geldorp, G. *Precise Orbit Computation, Gravity Model Adjustment and Altimeter Data Processing for the ERS-1 Altimetry Mission*,European Space Agency Contract Report, ESA Contract 6140/84/D/IM, Technisch Universiteit Delft, Delft, The Netherlands, Feb. 1987.