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OrekitTalk : Propagating Uncertainties

New methods of uncertainty propagation

09/21/2023 – Florian HUMEAU

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Introduction

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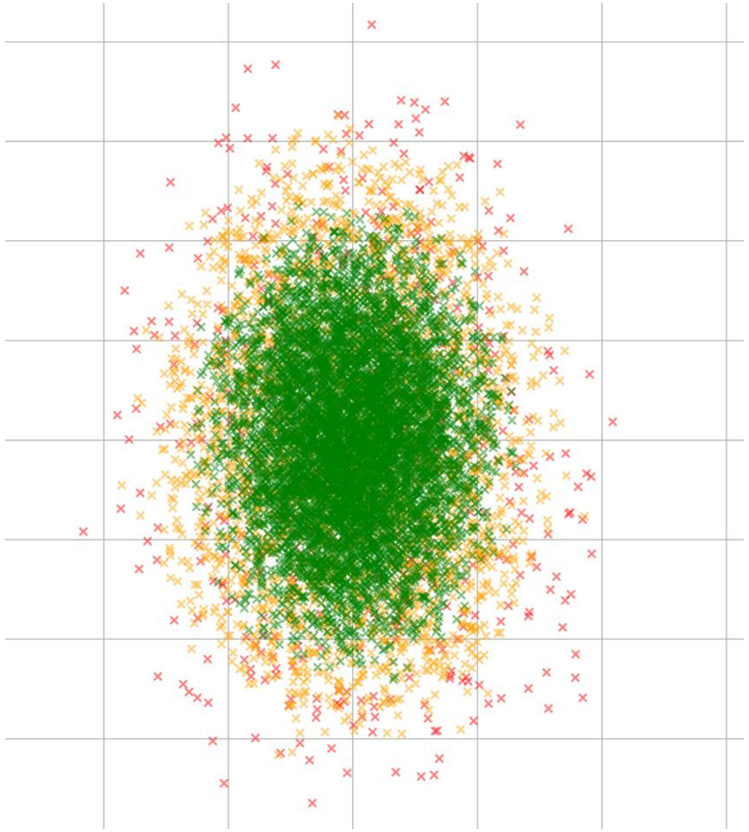


INTRODUCTION

Why does it matter?

- Needs to have a model for covariance AND uncertainty representation.
- Examples of applications could be for collision maneuvers.

Bold: Matrix or Vector

$\widetilde{Tilde}$: Propagated state

$\widehat{Circumflex}$: Approximated state

INTRODUCTION

Why does it matter?

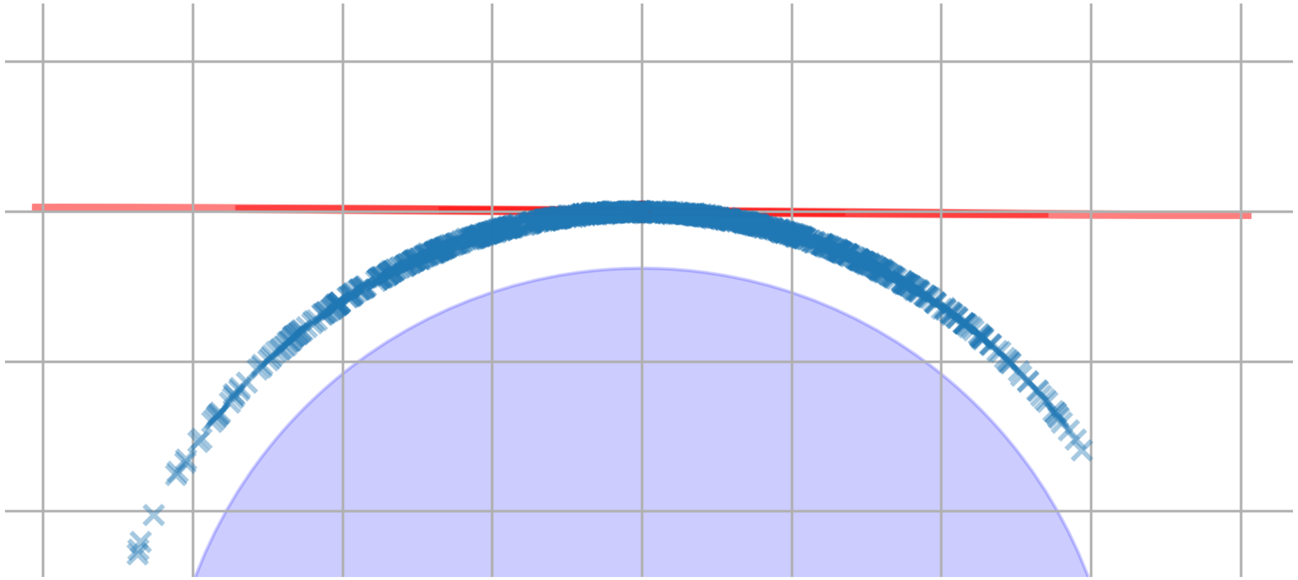
COVARIANCE REALISM ≠ UNCERTAINTY REALISM

- From a statistical point of view (68-95-99.7 law)
- Doesn’t take distribution into account

- Describe the distribution of the uncertainty
- Represent the exact “banana-shape”

What's currently done in Orekit

02



WHAT'S CURRENTLY DONE IN OREKIT

The method

- Today if you want to propagate the uncertainty of a spacecraft, you need to use a linear propagation:

Input : $\boldsymbol{\mu}$, $\boldsymbol{P}$ and $\boldsymbol{\Phi}$

- Propagation of the mean state :

$$\widetilde{\boldsymbol{\mu}} = \boldsymbol{\Phi}(\boldsymbol{\mu})$$

- Determination of the state transition matrix
- Determination of the propagated covariance

$$\widetilde{\boldsymbol{P}} = \boldsymbol{\phi}\boldsymbol{P}\boldsymbol{\phi}^{\boldsymbol{T}}$$

Output : $\widetilde{\boldsymbol{\mu}}$ and $\widetilde{\boldsymbol{P}}$

WHAT'S CURRENTLY DONE IN OREKIT

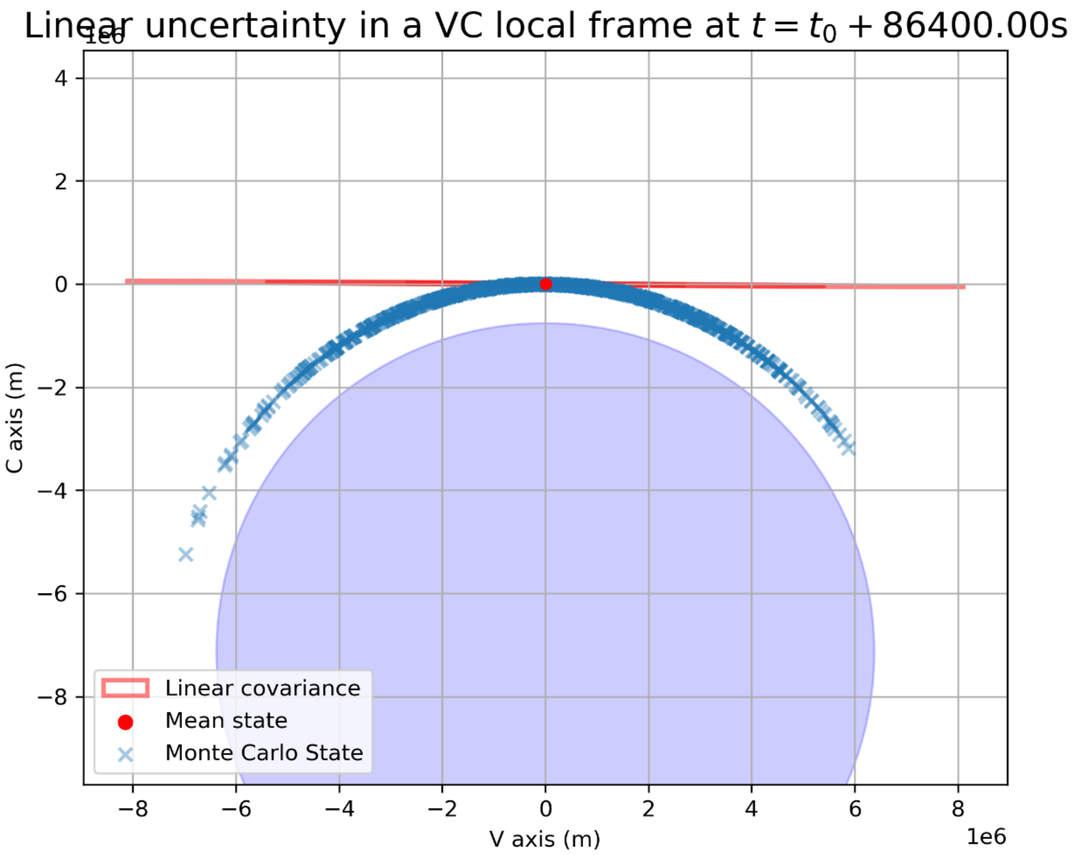
Results

The following test case if going to visualize the results of a given method. The dynamic is supposed to be Keplerian. This is an orbit with the following initial equinoctial conditions:

$$\boldsymbol{\mu} = \begin{bmatrix} 7136{,}635\ km \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ and } \boldsymbol{P} = \begin{bmatrix} (20\ km)^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 10^{-6} & 0 & 0 & 0 & 0 \\ 0 & 0 & 10^{-6} & 0 & 0 & 0 \\ 0 & 0 & 0 & 10^{-6} & 0 & 0 \\ 0 & 0 & 0 & 0 & 10^{-6} & 0 \\ 0 & 0 & 0 & 0 & 0 & (36'')^2 \end{bmatrix}$$

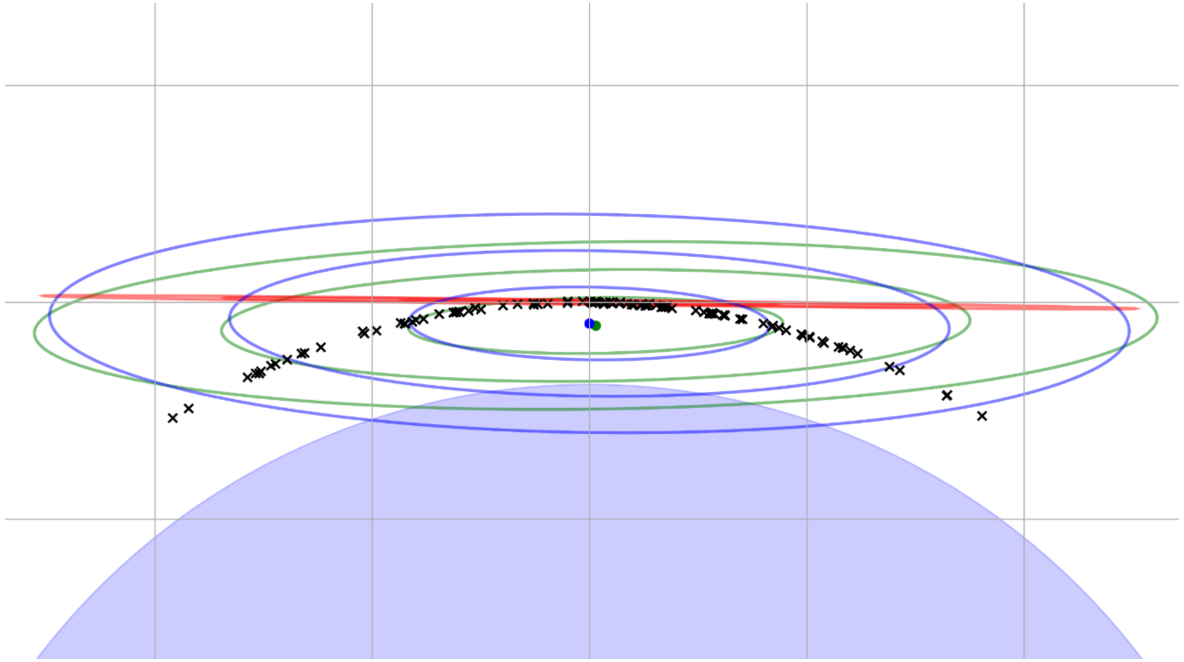
WHAT'S CURRENTLY DONE IN OREKIT

Results



Unscented Propagation

03



UNSCENTED PROPAGATION

Origin

Extracted from the Unscented Kalman Filter from Orekit and Hipparchus

UNSCENTED PROPAGATION

The method

Input : $\boldsymbol{\mu}$, $\boldsymbol{P}$ and $\boldsymbol{\Phi}$

- At $t = t_i$, compute $\forall k \in [\![1, n]\!]$:

$$\boldsymbol{\chi_0} = \boldsymbol{\mu}$$

$$\boldsymbol{\chi_k} = \boldsymbol{\mu} + \left(\sqrt{(n+\lambda)\boldsymbol{P}}\right)_k$$

$$\boldsymbol{\chi_{k+n}} = \boldsymbol{\mu} - \left(\sqrt{(n+\lambda)\boldsymbol{P}}\right)_k$$

13 Sigma points

With $\lambda = \alpha^2(n+\kappa) - n$

α the spread of the sigma points around the mean (e.g., $\approx 10^{-3}$)

κ a secondary scaling parameter (e.g., ≈ 0)

n the dimension of the state vector (here 6)

UNSCENTED PROPAGATION

The method

- Propagate the sigma points to $t = t_f \; \forall k \in \; [\![0, 2n]\!]$:

$$\widetilde{\boldsymbol{\chi}_{\boldsymbol{k}}} = \boldsymbol{\Phi}(\boldsymbol{\chi}_{\boldsymbol{k}})$$

- Reconstruct the mean and covariance of the propagated distribution using the propagated sigma points:

$$W_0^m = \frac{\lambda}{n+\lambda}$$

$$W_0^c = \frac{\lambda}{n+\lambda} + 1 - \alpha^2 + \beta$$

$$\forall k \in [\![1, 2n]\!], W_k^m = W_k^c = \frac{1}{2(n+\lambda)}$$

β used to incorporate prior knowledge of the distribution (e.g., ≈ 2 for gaussian)

UNSCENTED PROPAGATION

The method

Thus:

$$\widetilde{\boldsymbol{\mu}} = \sum_{k=0}^{2n} W_k^m \widetilde{\boldsymbol{\chi}_k}$$

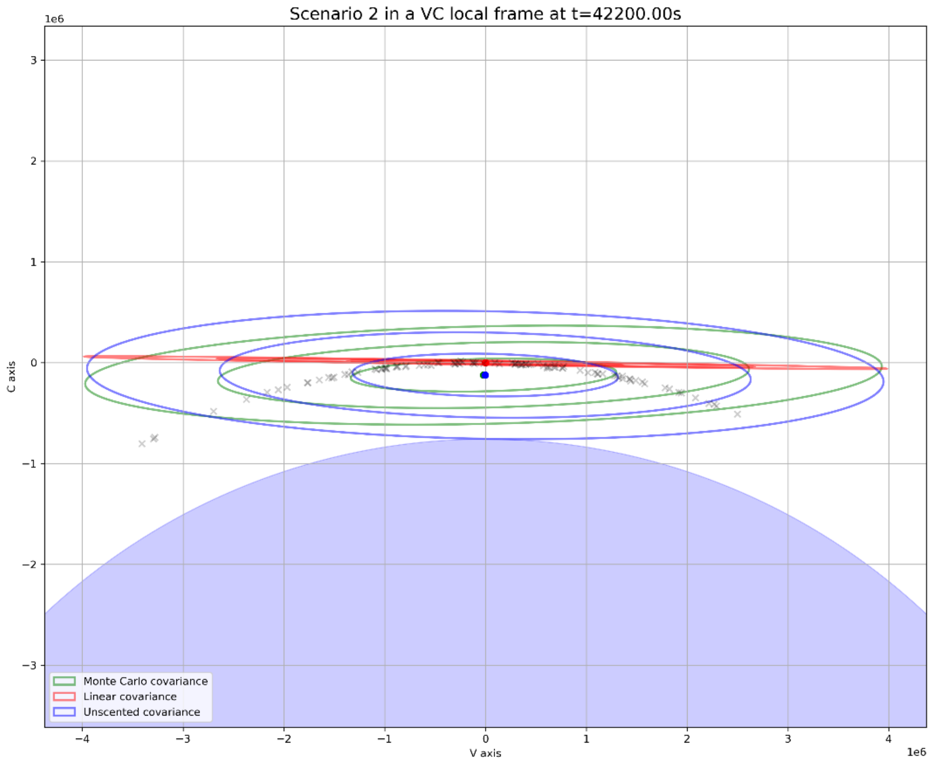
$$\widetilde{\boldsymbol{P}} = \sum_{k=0}^{2n} W_k^c (\widetilde{\boldsymbol{\chi}_k} - \widetilde{\boldsymbol{\mu}})(\widetilde{\boldsymbol{\chi}_k} - \widetilde{\boldsymbol{\mu}})^T + \boldsymbol{Q}$$

Output : $\widetilde{\boldsymbol{\mu}}$ and $\widetilde{\boldsymbol{P}}$

UNSCENTED PROPAGATION

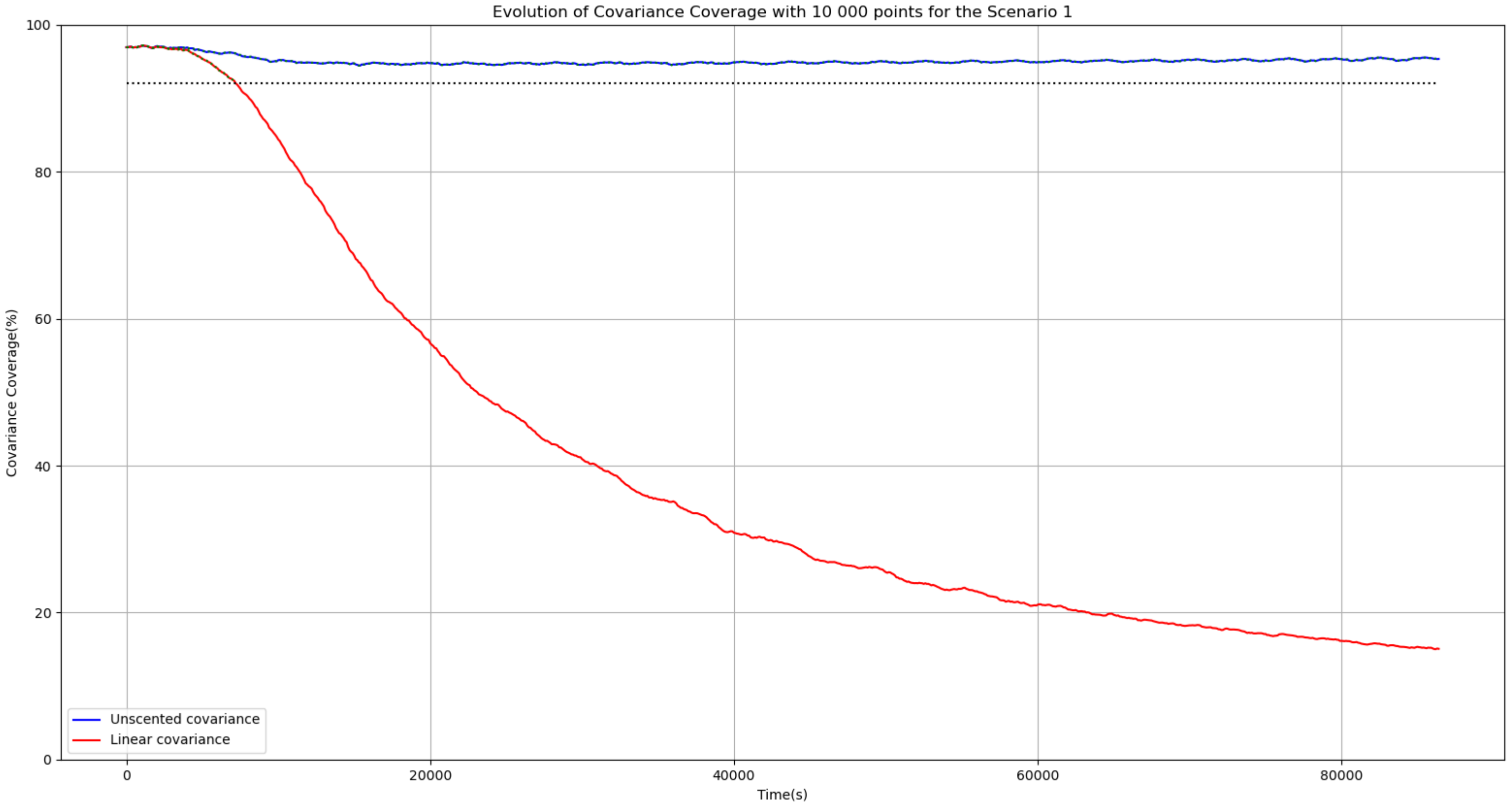
Results

Comparison between linear propagation, unscented propagation and Monte-Carlo reconstruction of the covariance.



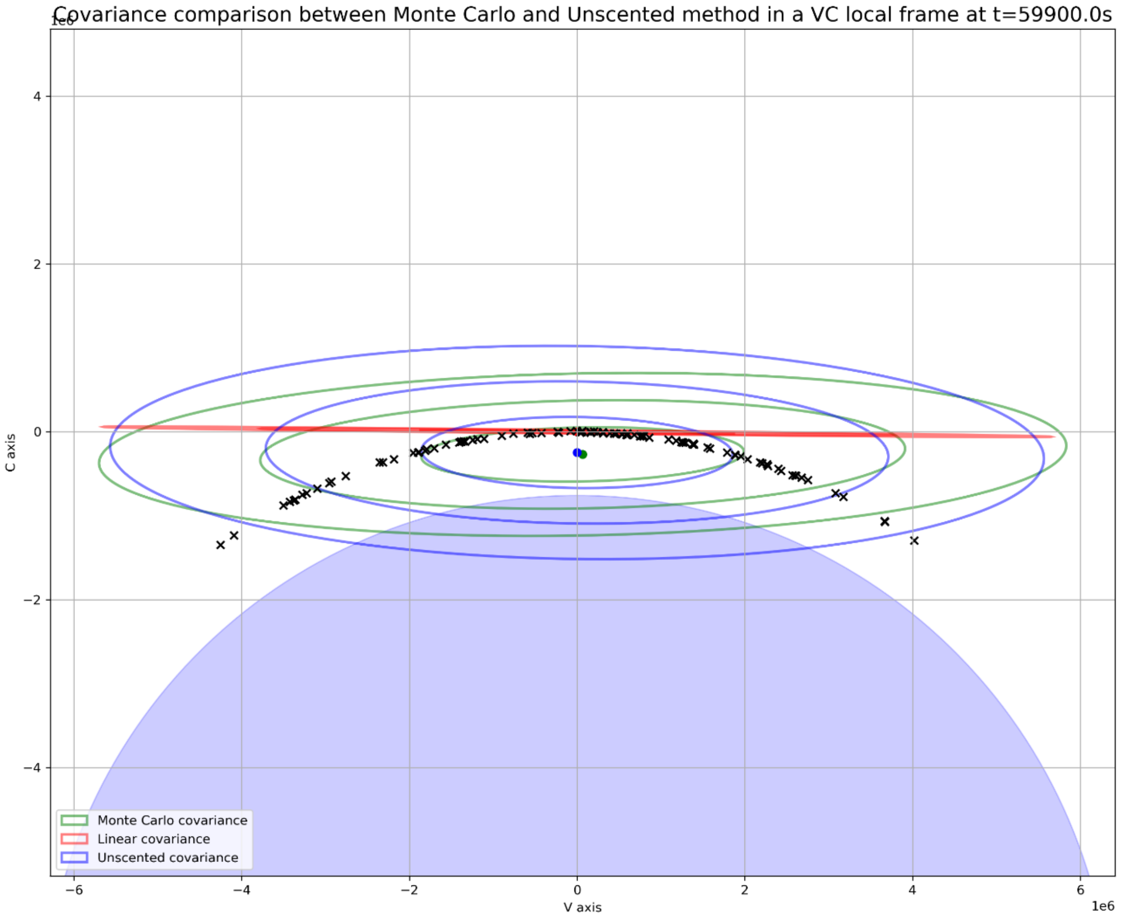
UNSCENTED PROPAGATION

Results



UNSCENTED PROPAGATION

Results

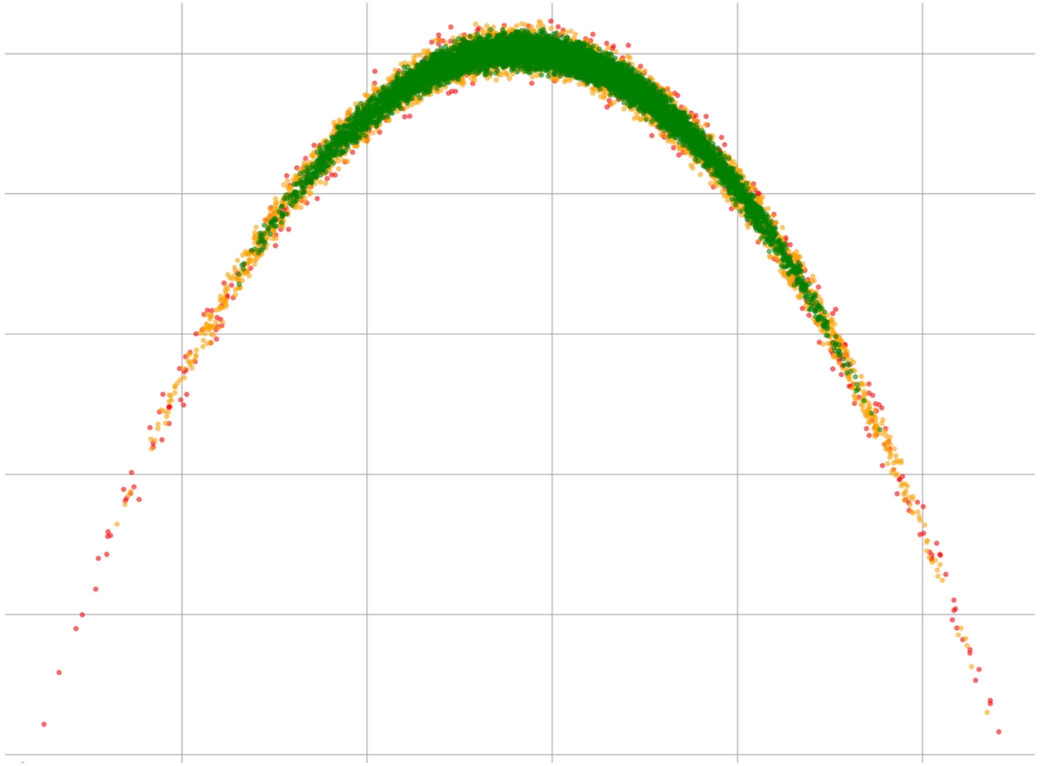


Mean state isn't on the orbit.

It doesn't represent the "mean state" of the spacecraft.

Gauss Von Mises Propagation

04



GAUSS VON MISES PROPAGATION

Origin

Based on the paper:

Joshua T. Horwood, Aubrey B. Poore, *“Gauss von Mises Distribution for Improved Uncertainty Realism in Space Situational Awareness”*

GAUSS VON MISES PROPAGATION

The method

The goal is to build a probability density function defining the uncertainty of the distribution. First, we define:

$$\boldsymbol{x} = \begin{bmatrix} a \\ e_x \\ e_y \\ h_x \\ h_y \end{bmatrix} \; and \; \theta = l_m$$

The method considers x to be represented by a gaussian distribution and θ by a Von Mises distribution.

Thus, the density of a specific point is:

$$p(\boldsymbol{x}, \theta) = N(\boldsymbol{x}; \boldsymbol{\mu}, \boldsymbol{P}) VM(\theta; \alpha, \boldsymbol{\beta}, \boldsymbol{\Gamma}, \kappa)$$

GAUSS VON MISES PROPAGATION

The method

Where:

$$N(x;\boldsymbol{\mu},\boldsymbol{P}) = \frac{1}{\sqrt{|2\pi\boldsymbol{P}|}} e^{-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^T\boldsymbol{P}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})}$$

$$VM(\theta;\Theta,\kappa) = \frac{1}{2\pi e^{-\kappa} I_0(\kappa)} e^{-2\kappa\sin^2\frac{1}{2}(\theta-\Theta)}$$

$$\Theta = \alpha + \boldsymbol{\beta}^T\boldsymbol{z} + \frac{1}{2}\boldsymbol{z}^T\boldsymbol{\Gamma}\boldsymbol{z}$$

$$\boldsymbol{z} = \boldsymbol{A}^{-1}(\boldsymbol{x}-\boldsymbol{\mu}), \qquad \boldsymbol{P} = \boldsymbol{A}\boldsymbol{A}^T$$

GAUSS VON MISES PROPAGATION

The method

The PDF has the following properties:

$p: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is a PDF on the cylinder $\mathbb{R}^n \times \mathbb{S}$

$p(\boldsymbol{x}, \theta) \geq 0$ almost everywhere on $\mathbb{R}^n \times \mathbb{R}$

$p(\boldsymbol{x}, \theta + 2\pi) = p(\boldsymbol{x}, \theta)$ almost everywhere on $\mathbb{R}^n \times \mathbb{R}$

$$\int_{\mathbb{R}^n} \int_{-\pi}^{\pi} p(\boldsymbol{x}, \theta)\, d\theta\, d\boldsymbol{x} = 1$$

GAUSS VON MISES PROPAGATION

The method

This method aims to propagate the parameters $\boldsymbol{\mu}$, $\boldsymbol{P}$, α, $\boldsymbol{\beta}$, $\boldsymbol{\Gamma}$ and κ.

Input : $\boldsymbol{\mu}$, $\boldsymbol{P}$ and $\boldsymbol{\Phi}$

- First, we need to extract the parameters from the gaussian equinoctial mean $\boldsymbol{x_{eq}}$ and covariance $\boldsymbol{P_{eq}}$:

$$\boldsymbol{x_{eq}} = \begin{bmatrix} \boldsymbol{\mu} \\ \alpha \end{bmatrix} \quad \boldsymbol{P_{eq}} = \begin{bmatrix} \boldsymbol{P} & \boldsymbol{A\beta} \\ \boldsymbol{\beta}^T \boldsymbol{A}^T & \boldsymbol{\beta}^T \boldsymbol{\beta} + \frac{1}{\kappa} \end{bmatrix}$$

With $\boldsymbol{P} = \boldsymbol{A}\boldsymbol{A}^T$ and $\boldsymbol{\Gamma} = 0$ for a Gaussian distribution.

GAUSS VON MISES PROPAGATION

The method

1. Sigma points

$$\begin{cases} \boldsymbol{N^{00}}\colon\ z = 0, \quad \phi = 0 \\ \boldsymbol{N^{\eta 0}}\colon\ z = 0, \quad |\phi| = \eta \\ \boldsymbol{N^{\xi 0}}\colon\ z_1 = \cdots = z_{i-1} = z_{i+1} = \cdots = z_n, \quad |z_i| = \xi, \quad \phi = 0 \end{cases}$$

13 Sigma points

With:

$$\xi = \sqrt{3}, \qquad \eta = \cos^{-1}\left(\frac{B_2(\kappa)}{2B_1(\kappa)} - 1\right)$$

And:

$$B_p(\kappa) = 1 - \frac{I_p(\kappa)}{I_0(\kappa)}$$

GAUSS VON MISES PROPAGATION

The method

Let's define $I_p(\kappa)$ as the modified Bessel function of the first kind of order p:

$$I_p(\kappa) = \frac{1}{\pi}\int_0^{\pi} e^{\kappa\cos(\theta)}\cos(p\theta)\,d\theta$$

As κ has a high magnitude, we can compute $e^{-\kappa} I_p(\kappa)$ in order to make calculations more accurate. Thus, we compute the asymptotic form for $\kappa \gg p$:

$$e^{-\kappa} I_p(\kappa) = \frac{1}{\sqrt{2\pi\kappa}}\left(1 + \sum_{i=1}^{\infty} \frac{\prod_{j=1}^{i} 4p^2 - (2j-1)^2}{i!\,(8\kappa)^i}(-1)^i\right)$$

GAUSS VON MISES PROPAGATION

The method

With that we can compute the weighted coefficient for each sigma points:

$$w_{\xi 0} = \frac{1}{6}, \qquad w_{\eta 0} = \frac{B_1(\kappa)^2}{4B_1(\kappa) - B_2(\kappa)}, \qquad w_{00} = 1 - 2w_{\eta 0} - 2nw_{\xi 0}$$

And determine the equinoctial coordinates using:

$$\boldsymbol{x}_{\boldsymbol{\sigma}_i} = \boldsymbol{\mu} + \boldsymbol{A}\boldsymbol{z}_{\boldsymbol{\sigma}_i}, \qquad \theta_{\sigma_i} = \phi_{\sigma_i} + \alpha + \boldsymbol{\beta}^T \boldsymbol{z}_{\boldsymbol{\sigma}_i} + \frac{1}{2}\boldsymbol{z}_{\boldsymbol{\sigma}_i}^T \boldsymbol{\Gamma} \boldsymbol{z}_{\boldsymbol{\sigma}_i}$$

We can finally propagate the sigma points:

$$\left(\widetilde{\boldsymbol{x}_{\boldsymbol{\sigma}_i}}, \widetilde{\theta_{\sigma_i}}\right) = \boldsymbol{\Phi}\left(\boldsymbol{x}_{\boldsymbol{\sigma}_i}, \theta_{\sigma_i}\right)$$

GAUSS VON MISES PROPAGATION

The method

2. Find $\widetilde{\boldsymbol{\mu}}$ and $\widetilde{\boldsymbol{P}}$

$$\widetilde{\boldsymbol{\mu}} = \sum_{i=1}^{2n+3} w_{\sigma_i} \widetilde{\boldsymbol{x}_{\boldsymbol{\sigma}_i}}, \qquad \widetilde{\boldsymbol{P}} = \sum_{i=1}^{2n+3} w_{\sigma_i} \left(\widetilde{\boldsymbol{x}_{\boldsymbol{\sigma}_i}} - \widetilde{\boldsymbol{\mu}}\right)\left(\widetilde{\boldsymbol{x}_{\boldsymbol{\sigma}_i}} - \widetilde{\boldsymbol{\mu}}\right)^T$$

And $\widetilde{\boldsymbol{P}} = \widetilde{\boldsymbol{A}}\widetilde{\boldsymbol{A}}^T$

3. Find the approximation $\hat{\alpha}, \widehat{\boldsymbol{\beta}}, \widehat{\boldsymbol{\Gamma}}$

$$\hat{\alpha} = \Phi_\theta(\boldsymbol{\mu}, \alpha)$$

$$\widehat{\boldsymbol{\beta}} = \widetilde{\boldsymbol{A}}^{-\boldsymbol{1}} \partial_x \boldsymbol{\Phi}_{\boldsymbol{x}}(\boldsymbol{\mu}, \alpha) \boldsymbol{A}[\boldsymbol{\beta} + \boldsymbol{A}^T \partial_x \boldsymbol{\Phi}_{\boldsymbol{\theta}}(\boldsymbol{\mu}, \alpha)]$$

$$\widehat{\boldsymbol{\Gamma}} = \widetilde{\boldsymbol{A}}^{-\boldsymbol{1}} \partial_x \boldsymbol{\Phi}_{\boldsymbol{x}}(\boldsymbol{\mu}, \alpha) \boldsymbol{A}[\boldsymbol{\Gamma} + \boldsymbol{A}^T \partial_x^2 \boldsymbol{\Phi}_{\boldsymbol{\theta}}(\boldsymbol{\mu}, \alpha) \boldsymbol{A}] \boldsymbol{A}^T \partial_x \boldsymbol{\Phi}_{\boldsymbol{x}}(\boldsymbol{\mu}, \alpha)^T \widetilde{\boldsymbol{A}}^{-\boldsymbol{T}}$$

The method

With the following terms for a two-body problem:

$$\boldsymbol{\partial_x \Phi_x}(\boldsymbol{\mu}, \alpha) \approx \boldsymbol{I}, \qquad \boldsymbol{\partial_x \Phi_\theta}(\boldsymbol{\mu}, \alpha) \approx \begin{bmatrix} -\frac{3}{2}\frac{n_0}{a_0}\Delta t \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad \boldsymbol{\partial_x^2 \Phi_\theta}(\boldsymbol{\mu}, \alpha) \approx \begin{bmatrix} \frac{15}{4}\frac{n_0}{a_0^2}\Delta t & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Where:

$$a_0 = \mu_1, \qquad n_0 = \sqrt{\frac{GM}{a_0^3}}, \qquad \Delta t = t_f - t_i$$

The method

4. Find corrected $\tilde{\alpha}, \widetilde{\boldsymbol{\beta}}$ and $\widetilde{\boldsymbol{\Gamma}}$

The goal is to solve a least square problem as follows knowing $\tilde{\kappa} = \kappa$:

$$(\tilde{\alpha}, \widetilde{\boldsymbol{\beta}}, \widetilde{\boldsymbol{\Gamma}}) = \underset{\hat{\alpha}, \widehat{\boldsymbol{\beta}}, \widehat{\boldsymbol{\Gamma}}}{\arg\min} \sum_{i=1}^{2n+3} \left[r_i(\hat{\alpha}, \widehat{\boldsymbol{\beta}}, \widehat{\boldsymbol{\Gamma}})\right]^2$$

With the residual function defined as:

$$r_i(\hat{\alpha}, \widehat{\boldsymbol{\beta}}, \widehat{\boldsymbol{\Gamma}}) = \boldsymbol{z}_{\boldsymbol{\sigma_i}}^T \boldsymbol{z}_{\boldsymbol{\sigma_i}} - \widetilde{\boldsymbol{z}_{\boldsymbol{\sigma_i}}}^T \widetilde{\boldsymbol{z}_{\boldsymbol{\sigma_i}}} + 4\kappa \left(\left(\sin \frac{1}{2} \phi_{\sigma_i} \right)^2 - \left(\sin \frac{1}{2} \widetilde{\phi_{\sigma_i}} \right)^2 \right)$$

$$\widetilde{\boldsymbol{z}_{\boldsymbol{\sigma_i}}} = \widetilde{\boldsymbol{A}}^{-1} \left(\widetilde{\boldsymbol{x}_{\boldsymbol{\sigma_i}}} - \widetilde{\boldsymbol{\mu}} \right)$$

$$\widetilde{\phi_{\sigma_i}} = \widetilde{\theta_{\sigma_i}} - \hat{\alpha} - \widehat{\boldsymbol{\beta}}^T \widetilde{\boldsymbol{z}_{\boldsymbol{\sigma_i}}} - \frac{1}{2} \widetilde{\boldsymbol{z}_{\boldsymbol{\sigma_i}}}^T \widehat{\boldsymbol{\Gamma}} \widetilde{\boldsymbol{z}_{\boldsymbol{\sigma_i}}}$$

GAUSS VON MISES PROPAGATION

The method

And the partial derivatives of the residual function for a two-body problem:

$$\frac{\partial r_i}{\partial \hat{\alpha}} = 2\kappa \sin \widetilde{\phi_{\sigma_i}}, \qquad \frac{\partial r_i}{\partial \widehat{\boldsymbol{\beta}}} = 2\kappa \sin \widetilde{\phi_{\sigma_i}}\, \widetilde{\boldsymbol{z}_{\boldsymbol{\sigma}_i}}, \qquad \frac{\partial r_i}{\partial \widehat{\Gamma_{11}}} = \kappa \sin \widetilde{\phi_{\sigma_i}} \left[\left(\widetilde{\boldsymbol{z}_{\boldsymbol{\sigma}_i}}\right)_{11}\right]^2$$

Thus, we finally can compute:

$$p\widetilde{(\boldsymbol{x},\theta)} = GVM\left(\boldsymbol{x},\theta;\widetilde{\boldsymbol{\mu}},\widetilde{\boldsymbol{P}},\tilde{\alpha},\widetilde{\boldsymbol{\beta}},\widetilde{\boldsymbol{\Gamma}},\tilde{\kappa}\right)$$

Output : A Gauss von Mises probability density function

Averaged Uncertainty Realism Metric

05

AVERAGED UNCERTAINTY REALISM METRIC

The method

Metric to accurately represent the uncertainty realism of the pdfs in a statistic sense, using the Mahalanobis distance of propagated points in this pdf:

Thus:

$$\bar{U} = \frac{1}{nk}\sum_{i=1}^{k} U_i$$

With U_i the mahalanobis distance of point i,

n the number of parameters (here $n = 6$),

k the number of propagated points,

AVERAGED UNCERTAINTY REALISM METRIC

The method

For a Gauss Von Mises distribution, we have:

$$U_i(\boldsymbol{x_i}, \theta_i; \boldsymbol{\mu}, \boldsymbol{P}, \alpha, \boldsymbol{\beta}, \boldsymbol{\Gamma}, \kappa) = (\boldsymbol{x_i} - \boldsymbol{\mu})^T \boldsymbol{P}^{-1} (\boldsymbol{x_i} - \boldsymbol{\mu}) + 4\kappa \sin^2 \frac{1}{2} (\theta_i - \Theta(\boldsymbol{x_i}))$$

And for a normal multivariate distribution:

$$U_i(\boldsymbol{x_i}; \boldsymbol{\mu}, \boldsymbol{P}) = (\boldsymbol{x_i} - \boldsymbol{\mu})^T \boldsymbol{P}^{-1} (\boldsymbol{x_i} - \boldsymbol{\mu})$$

AVERAGED UNCERTAINTY REALISM METRIC

The method

Under weak assumptions, the computed test statistic is approximately chi-squared distributed. Thus, we can find the 99.9% interval of the distribution:

$$\frac{1}{nk}\chi^2(nk)$$

And find when the metric steps out of this area.

Normalized Averaged Density Metric

06

NORMALIZED AVERAGED DENSITY METRIC

The method

As we've seen, the first metric only accounts for the statistical uncertainty of the distribution, the thus need to emphasize the actual uncertainty realism. We settled of the normalized averaged density metric

Thus, the averaged density metric is:

$$\widetilde{D(t)} = \frac{1}{k}\sum_{i=1}^{k} D_i(t)$$

And the normalized averaged density metric is:

$$\overline{D(t)} = \frac{\widetilde{D(t)}}{\widetilde{D(t_0)}}$$

NORMALIZED AVERAGED DENSITY METRIC

The method

For a Gauss von Mises distribution the density is computed as:

$$D(\boldsymbol{x}, \theta) = N(\boldsymbol{x}; \boldsymbol{\mu}, \boldsymbol{P}) VM(\theta; \alpha, \boldsymbol{\beta}, \boldsymbol{\Gamma}, \kappa)$$

And for a normal multivariate distribution:

$$D(\boldsymbol{x}, \theta) = N(\boldsymbol{x}; \boldsymbol{\mu}, \boldsymbol{P})$$

NORMALIZED AVERAGED DENSITY METRIC

The method

With:

$$N(x;\boldsymbol{\mu},\boldsymbol{P}) = \frac{1}{\sqrt{|2\pi\boldsymbol{P}|}} e^{-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^T\boldsymbol{P}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})}$$

$$VM(\theta;\Theta,\kappa) = \frac{1}{2\pi e^{-\kappa} I_0(\kappa)} e^{-2\kappa \sin^2 \frac{1}{2}(\theta-\Theta)}$$

$$\Theta = \alpha + \boldsymbol{\beta}^T\boldsymbol{z} + \frac{1}{2}\boldsymbol{z}^T\boldsymbol{\Gamma}\boldsymbol{z}$$

$$\boldsymbol{z} = \boldsymbol{A}^{-1}(\boldsymbol{x}-\boldsymbol{\mu}), \qquad \boldsymbol{P} = \boldsymbol{A}\boldsymbol{A}^T$$

Test Cases

07

TEST CASES

Origin

Based on the paper:

J. T. Horwood, J. M. Aristoff, N. Singh, and A. B. Poore, *"A comparative study of new non-linear uncertainty propagation methods for space surveillance,"* in Proceedings of the SPIE, Signal and Data Processing of Small Targets, Vol. 9092, Baltimore, MD, May 2014

TEST CASES

Covariances

For the test cases, we're going to use 3 levels of covariances.

Low accuracy covariance:

$$\boldsymbol{P} = \begin{bmatrix} (20\ km)^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 10^{-6} & 0 & 0 & 0 & 0 \\ 0 & 0 & 10^{-6} & 0 & 0 & 0 \\ 0 & 0 & 0 & 10^{-6} & 0 & 0 \\ 0 & 0 & 0 & 0 & 10^{-6} & 0 \\ 0 & 0 & 0 & 0 & 0 & (36")^2 \end{bmatrix}$$

TEST CASES

Covariances

Medium accuracy covariance:

$$
\boldsymbol{P} = \begin{bmatrix} (2\ km)^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 10^{-8} & 0 & 0 & 0 & 0 \\ 0 & 0 & 10^{-8} & 0 & 0 & 0 \\ 0 & 0 & 0 & 10^{-8} & 0 & 0 \\ 0 & 0 & 0 & 0 & 10^{-8} & 0 \\ 0 & 0 & 0 & 0 & 0 & (28")^2 \end{bmatrix}
$$

TEST CASES

Covariances

High accuracy covariance:

$$\boldsymbol{P} = \begin{bmatrix} (50\,m)^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 10^{-10} & 0 & 0 & 0 & 0 \\ 0 & 0 & 10^{-10} & 0 & 0 & 0 \\ 0 & 0 & 0 & 10^{-10} & 0 & 0 \\ 0 & 0 & 0 & 0 & 10^{-10} & 0 \\ 0 & 0 & 0 & 0 & 0 & (20'')^2 \end{bmatrix}$$

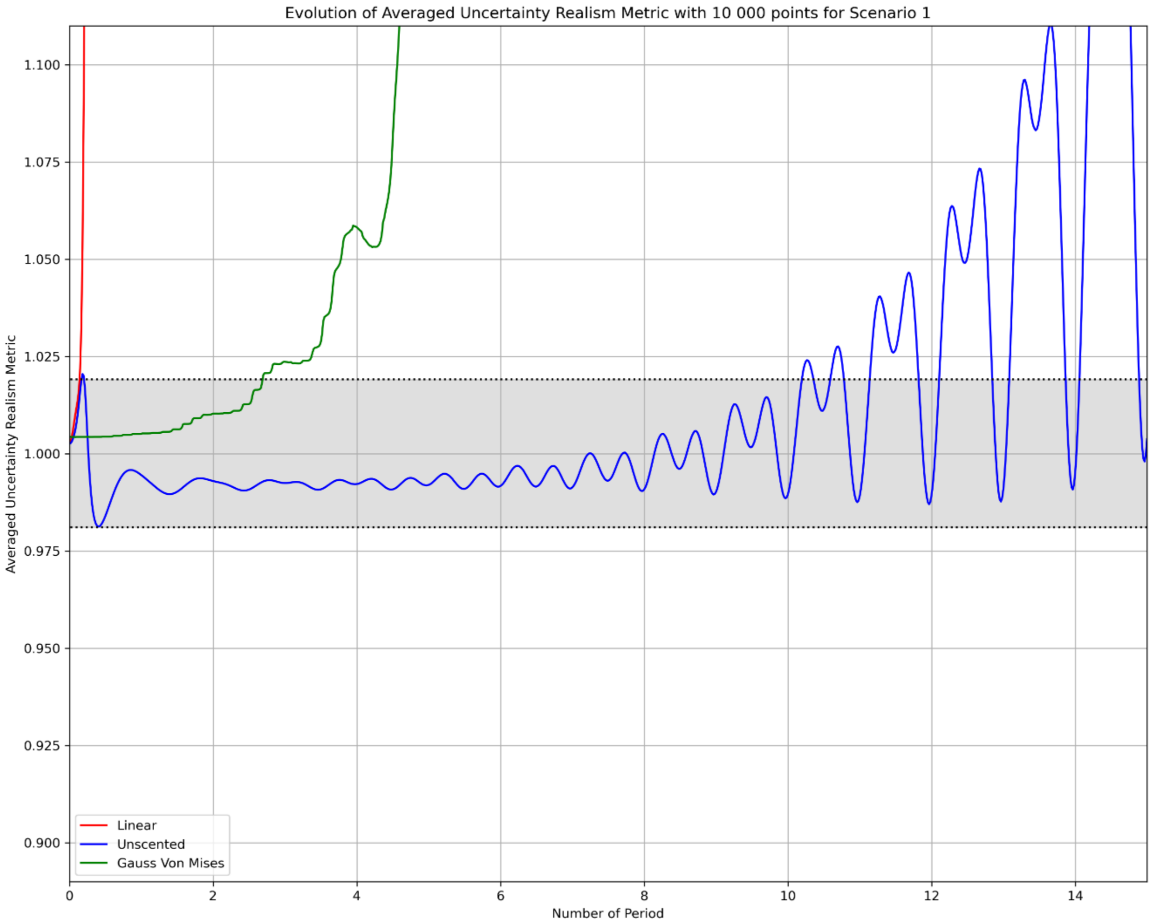
Scenarios

And using the following scenarios:

Scenario	Orbit Type	Orbit Accuracy	a(km)	e	i(°)	Ω(°)	ω(°)	M(°)	Timespan (days)
1	LEO	Low	7136.6	0.00949	72.9	116.0	57.7	105.5	1
2	LEO	Low	7136.6	0.0	0.0	0.0	0.0	0.0	1
3	LEO	High	7136.6	0.00949	72.9	116.0	57.7	105.5	7
4	GEO	Medium	42164.1	0.0	0.0	0.0	0.0	0.0	30
5	HEO	Medium	26628.1	0.742	63.4	120.0	0.0	144.0	7

Results

08



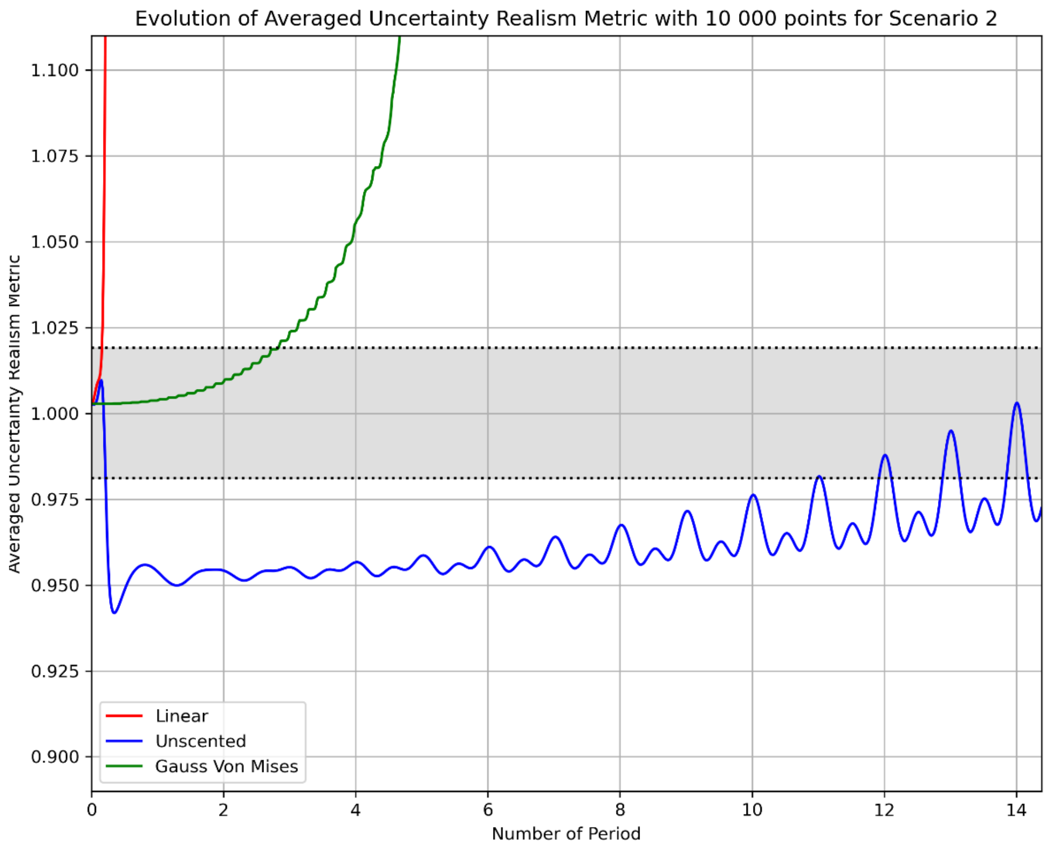
Evolution of Averaged Uncertainty Realism Metric with 10 000 points for Scenario 1
Averaged Uncertainty Realism Metric
Number of Period
1.100
1.075
1.050
1.025
1.000
0.975
0.950
0.925
0.900
0
2
4
6
8
10
12
14
Linear
Unscented
Gauss Von Mises

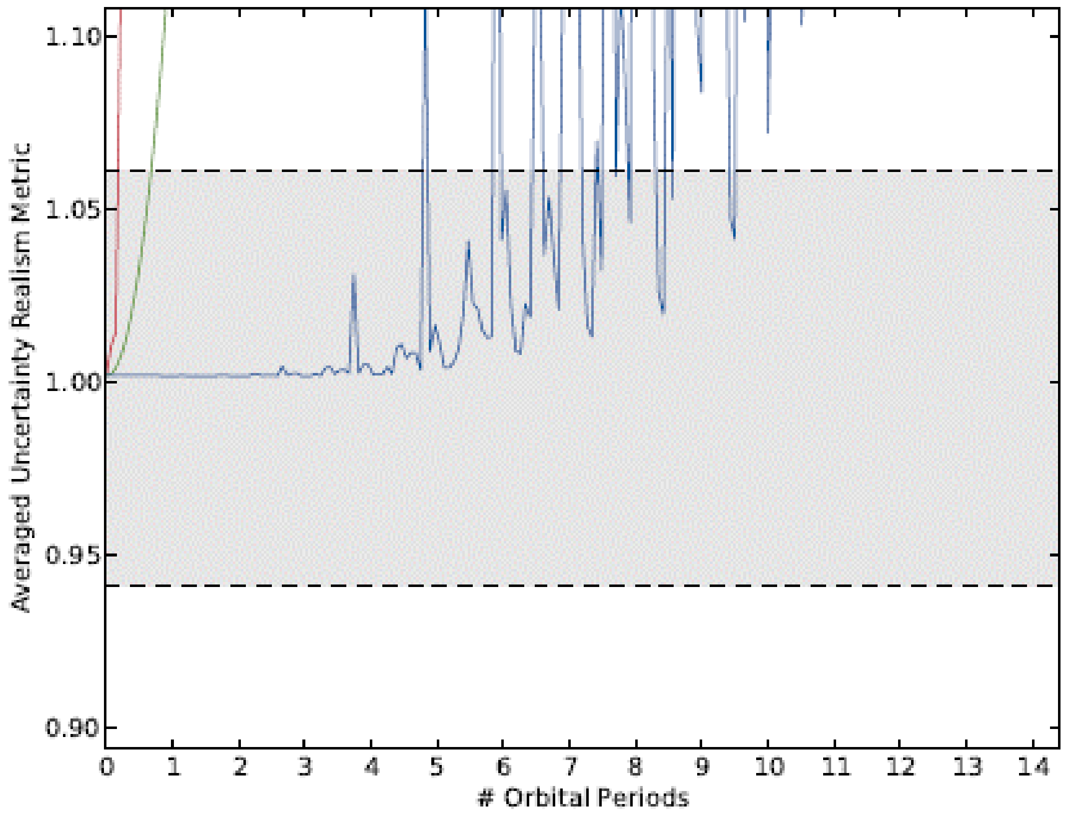
Scenario	Orbit Type	Orbit Accuracy	a(km)	e	i(°)	Ω(°)	ω(°)	M(°)	Timespan (days)
2	LEO	Low	7136.6	0.0	0.0	0.0	0.0	0.0	1

RESULTS

Scenario 2 – Linear

Focus on the red curve





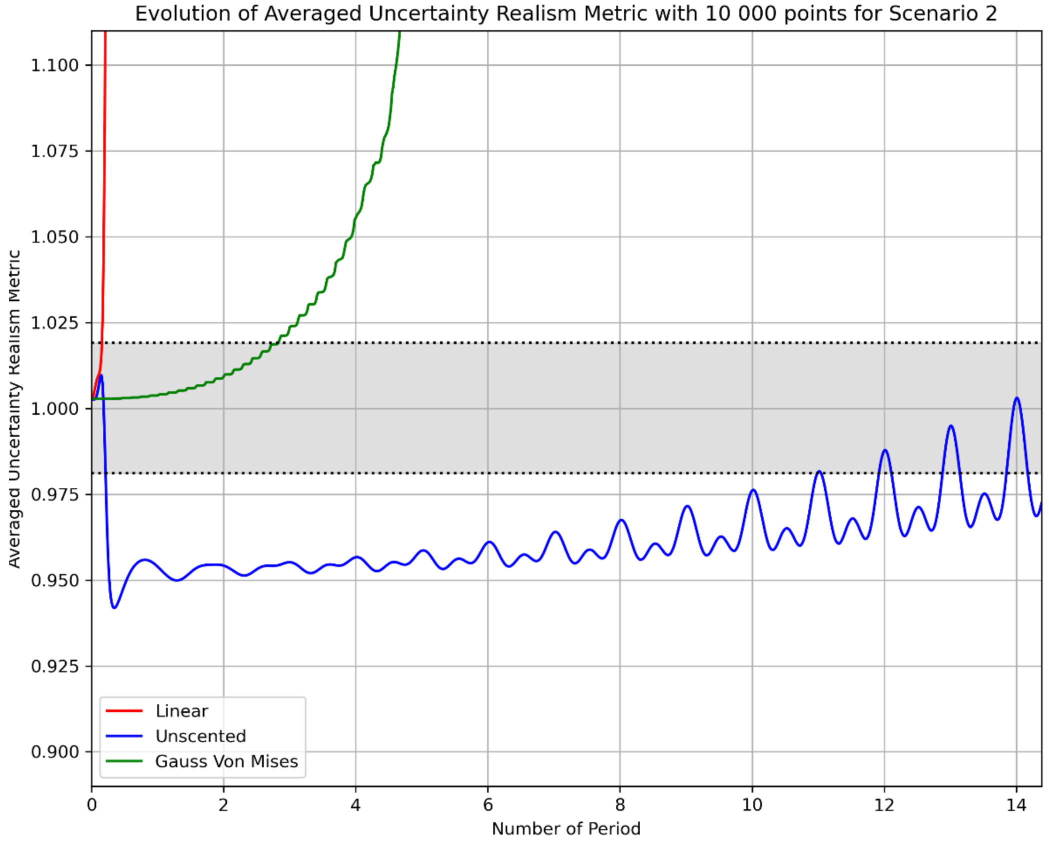
Credit: A comparative study of new non-linear uncertainty propagation methods for space surveillance

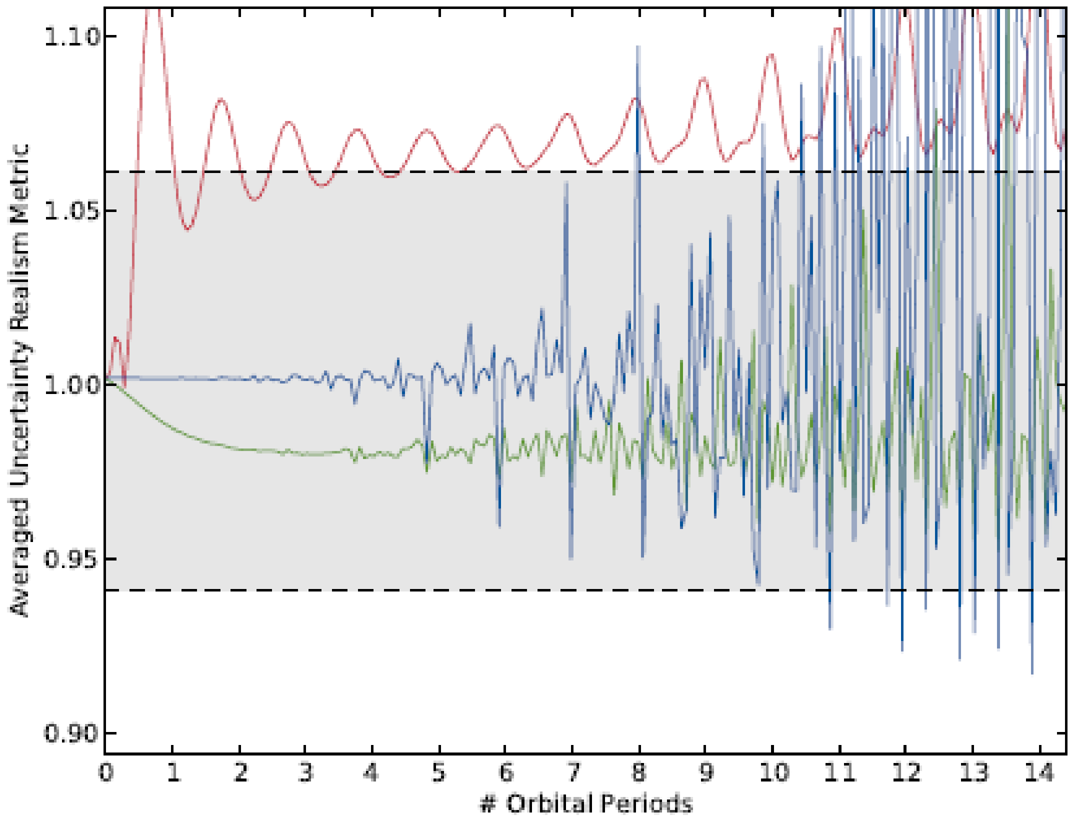
RESULTS

Scenario 2 – Unscented

Scenario	Orbit Type	Orbit Accuracy	a(km)	e	i(°)	Ω(°)	ω(°)	M(°)	Timespan (days)
2	LEO	Low	7136.6	0.0	0.0	0.0	0.0	0.0	1

Focus on the red curve





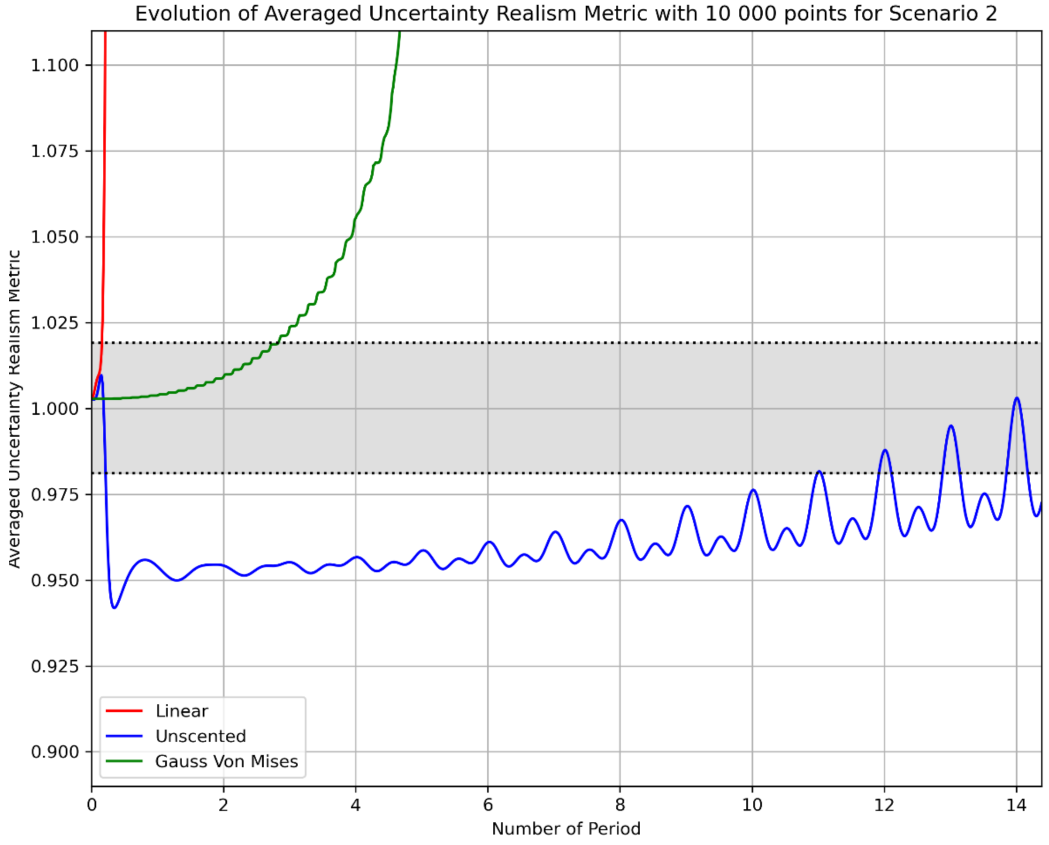
Credit: A comparative study of new non-linear uncertainty propagation methods for space surveillance

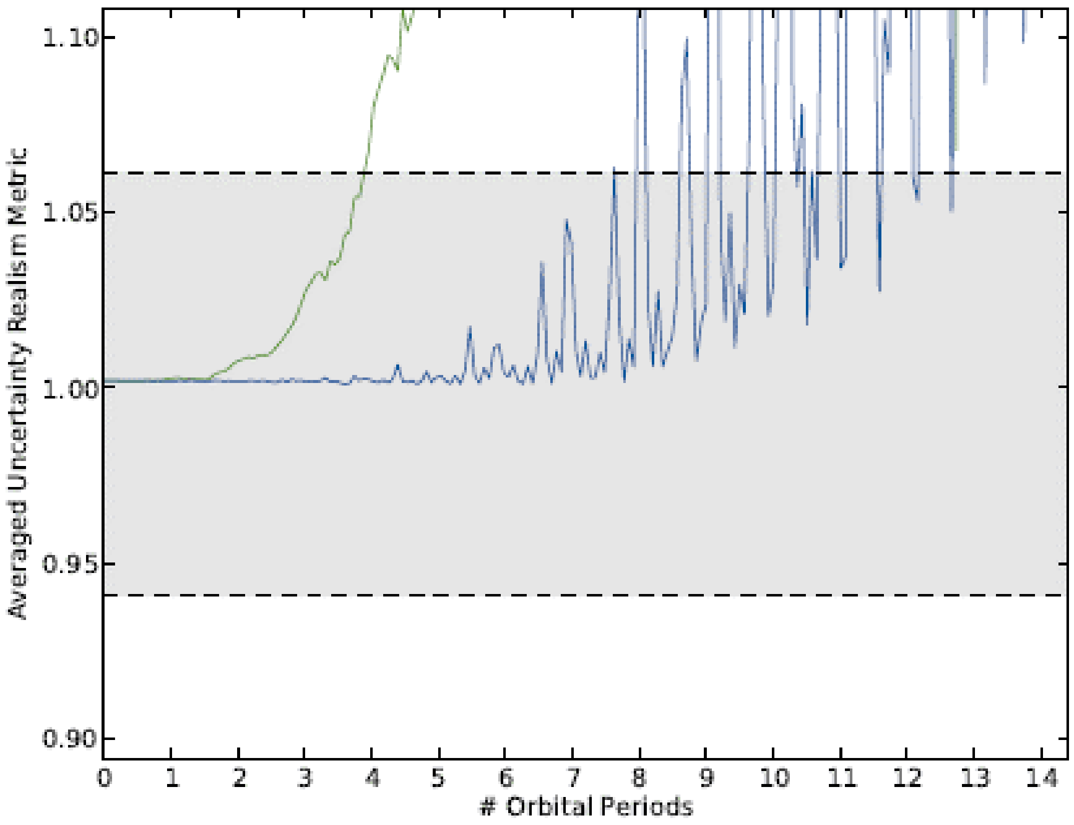
Scenario	Orbit Type	Orbit Accuracy	a(km)	e	i(°)	Ω(°)	ω(°)	M(°)	Timespan (days)
2	LEO	Low	7136.6	0.0	0.0	0.0	0.0	0.0	1

RESULTS

Scenario 2 – Gauss Von Mises

Focus on the green curve



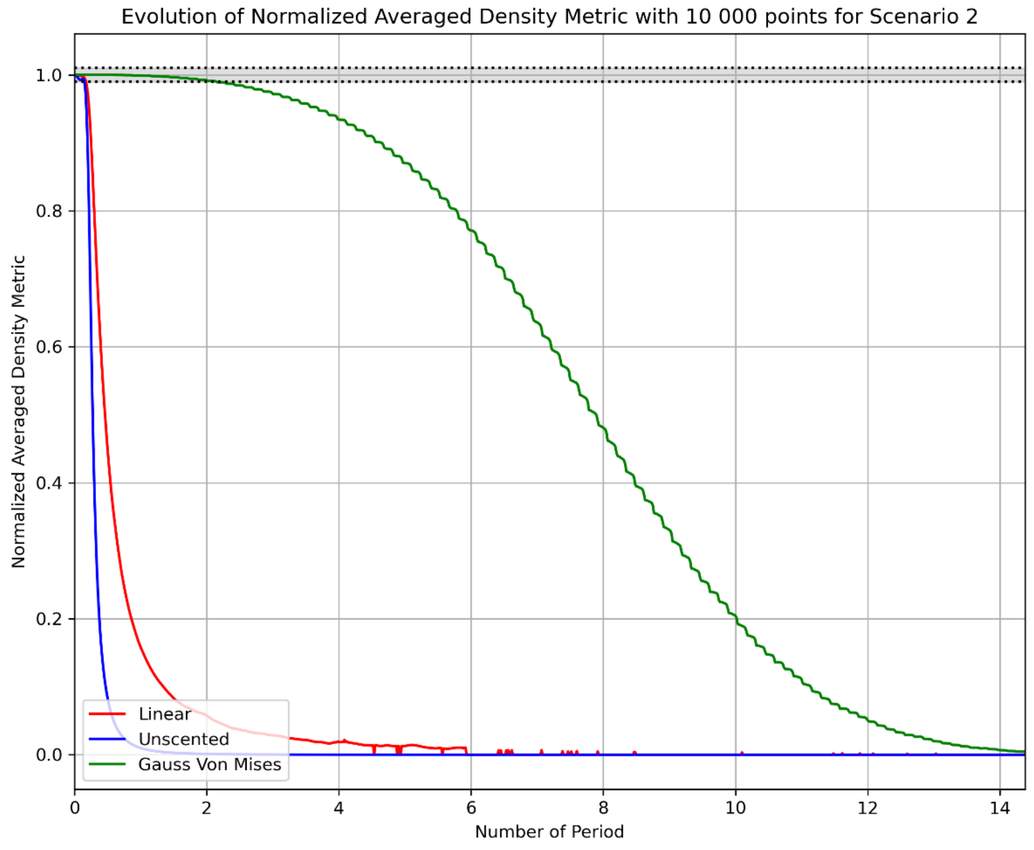


Credit: A comparative study of new non-linear uncertainty propagation methods for space surveillance

Scenario	Orbit Type	Orbit Accuracy	a(km)	e	i(°)	Ω(°)	ω(°)	M(°)	Timespan (days)
2	LEO	Low	7136.6	0.0	0.0	0.0	0.0	0.0	1

RESULTS

Scenario 2



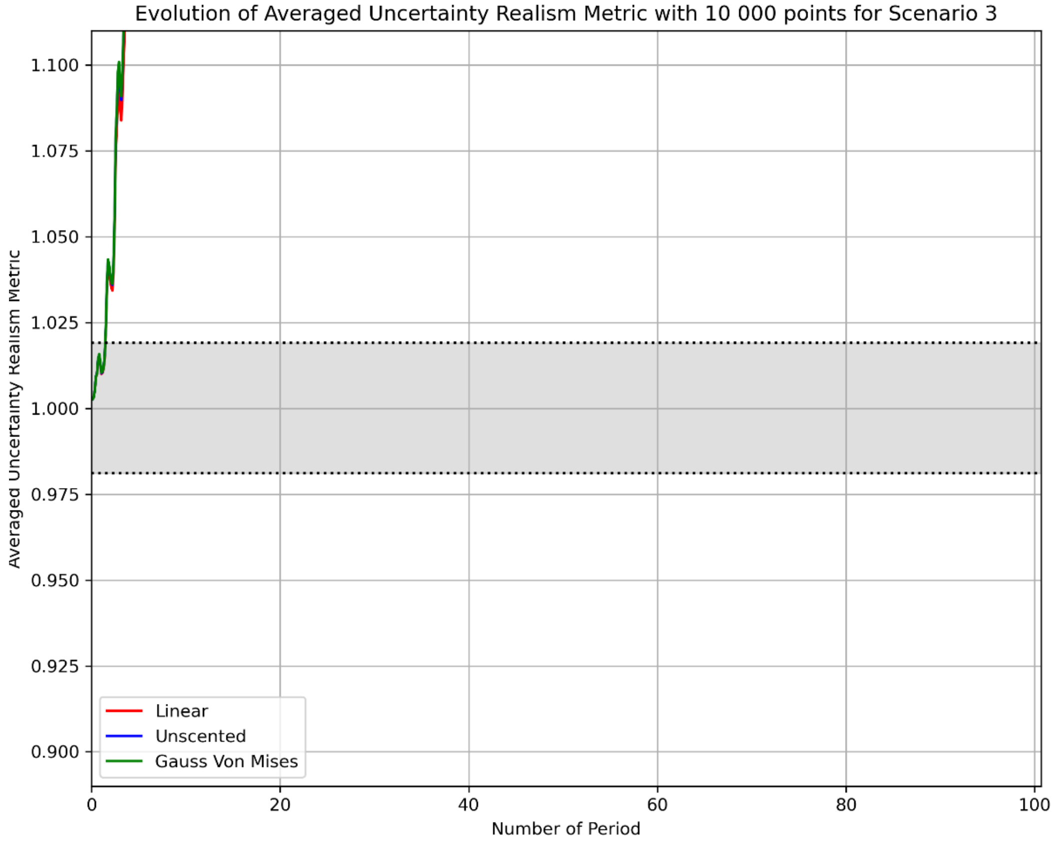
	NADM (orbit)
Linear	0.188
Unscented	0.147
Gauss Von Mises	2.223

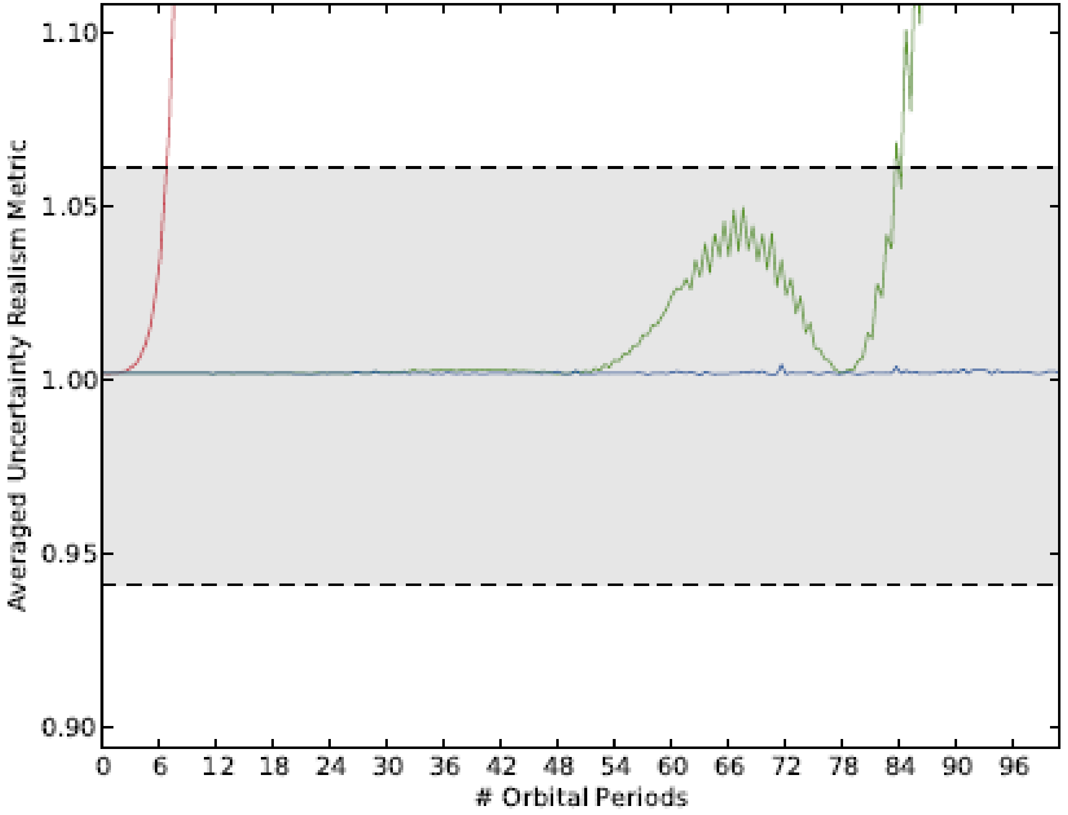
Scenario	Orbit Type	Orbit Accuracy	a(km)	e	i(°)	Ω(°)	ω(°)	M(°)	Timespan (days)
3	LEO	High	7136.6	0.00949	72.9	116.0	57.7	105.5	7

RESULTS

Scenario 3 – Linear

Focus on the red curve





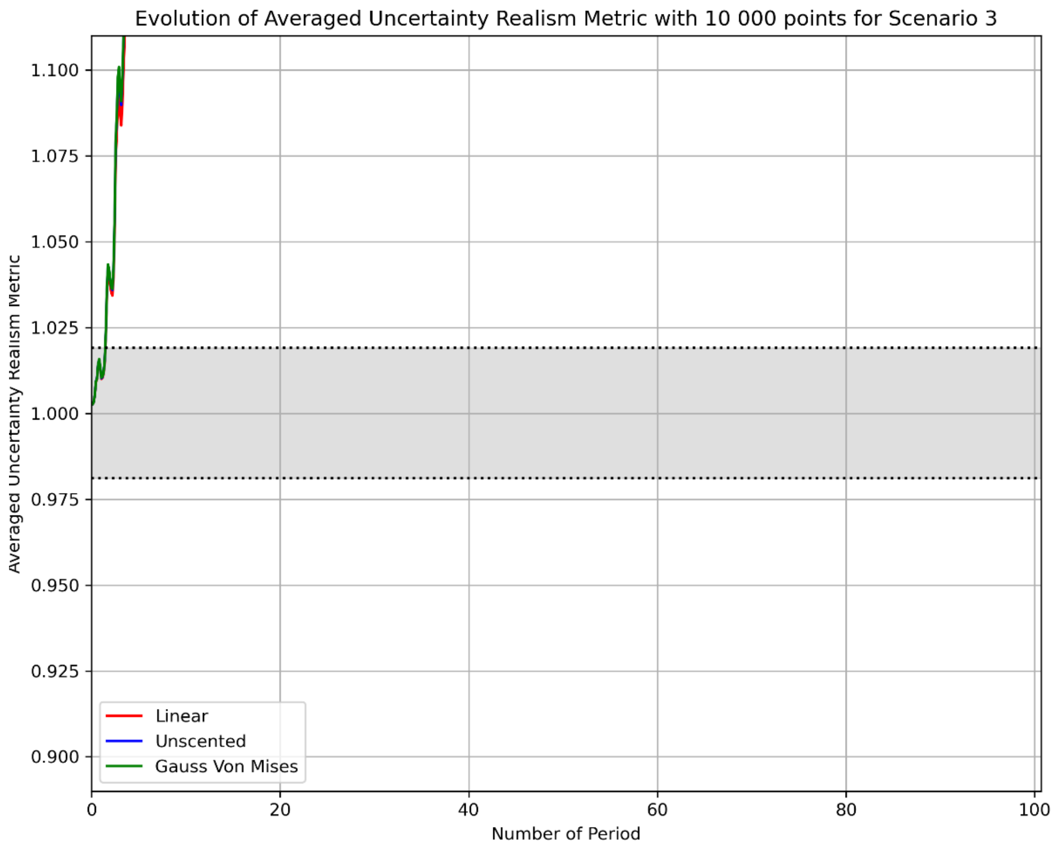
Credit: A comparative study of new non-linear uncertainty propagation methods for space surveillance

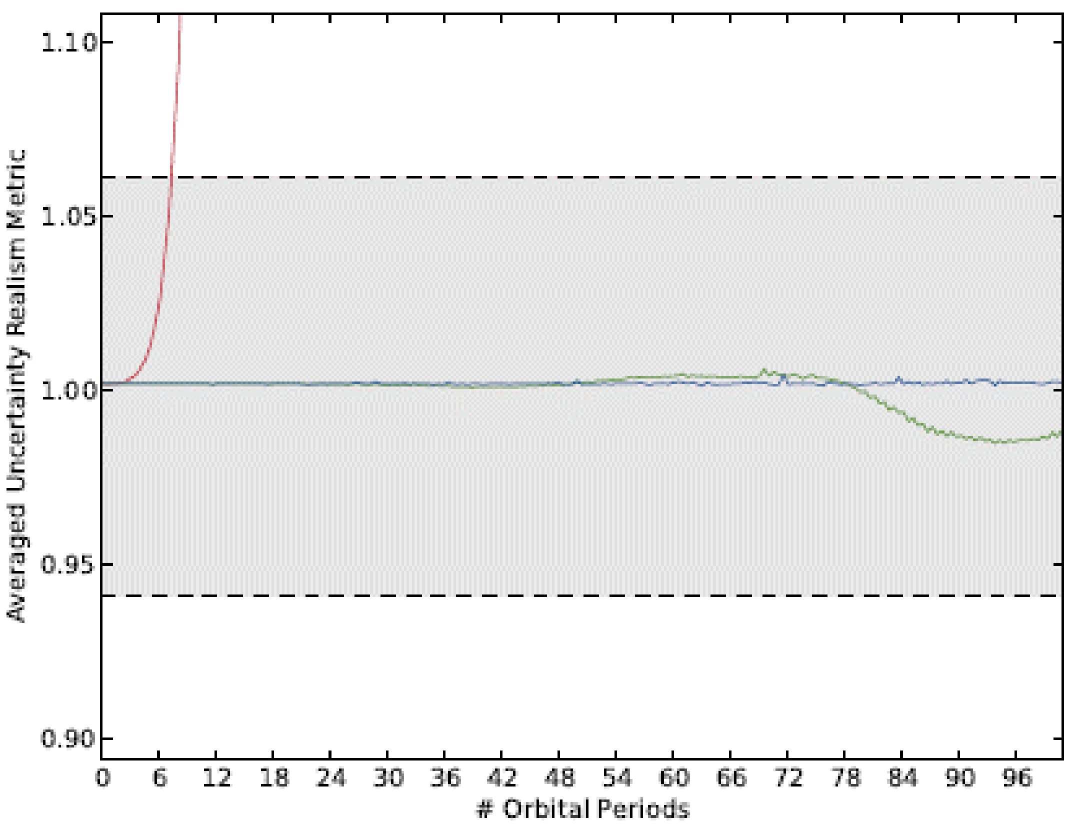
Scenario	Orbit Type	Orbit Accuracy	a(km)	e	i(°)	Ω(°)	ω(°)	M(°)	Timespan (days)
3	LEO	High	7136.6	0.00949	72.9	116.0	57.7	105.5	7

RESULTS

Scenario 3 – Unscented

Focus on the red curve





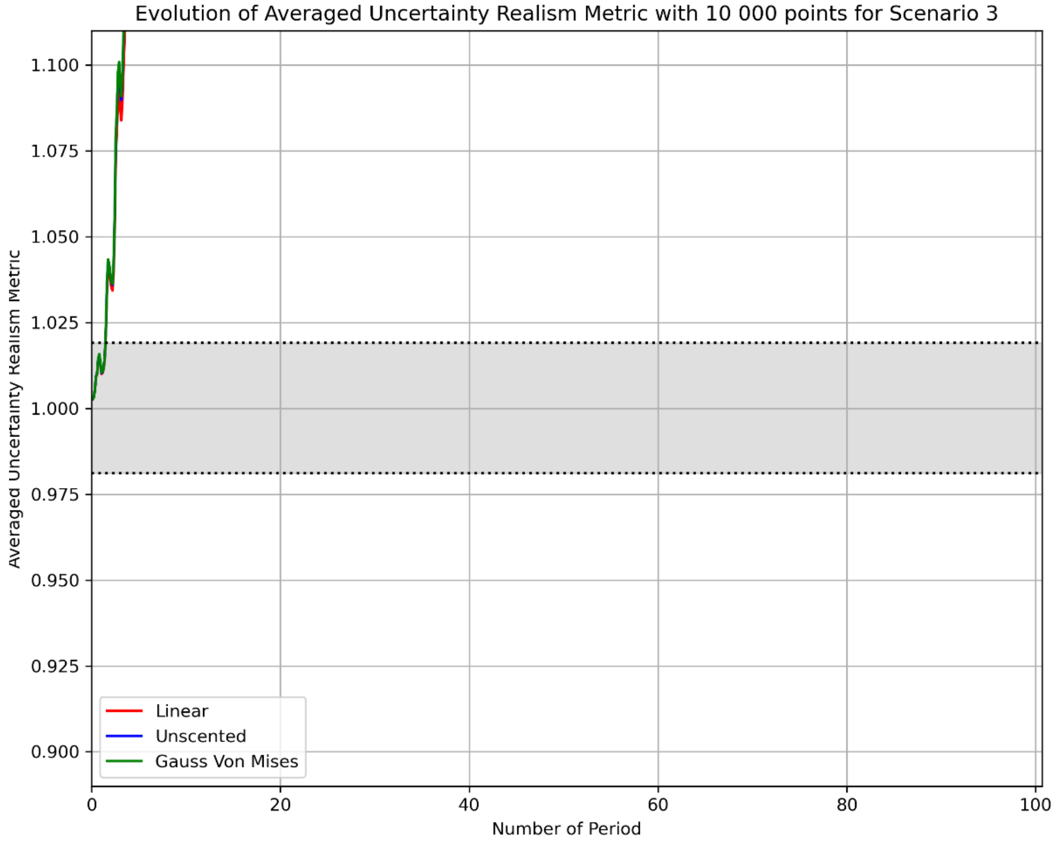
Credit: A comparative study of new non-linear uncertainty propagation methods for space surveillance

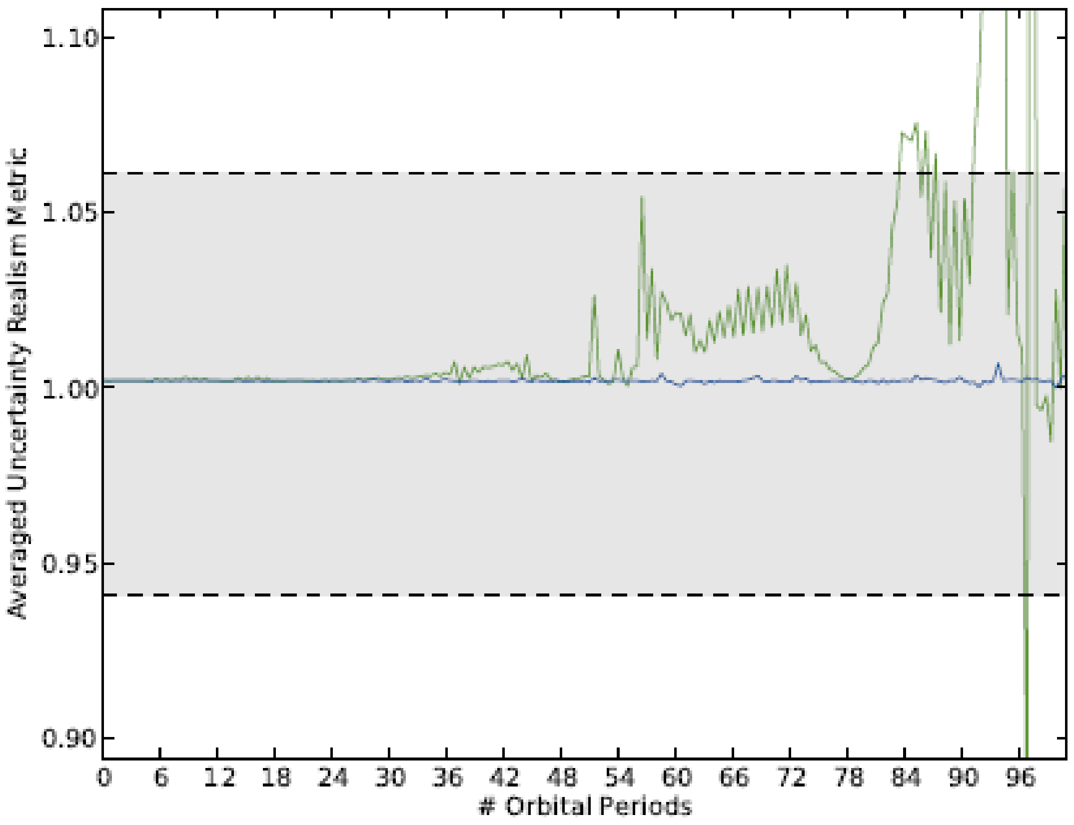
Scenario	Orbit Type	Orbit Accuracy	a(km)	e	i(°)	Ω(°)	ω(°)	M(°)	Timespan (days)
3	LEO	High	7136.6	0.00949	72.9	116.0	57.7	105.5	7

RESULTS

Scenario 3 – Gauss Von Mises

Focus on the green curve



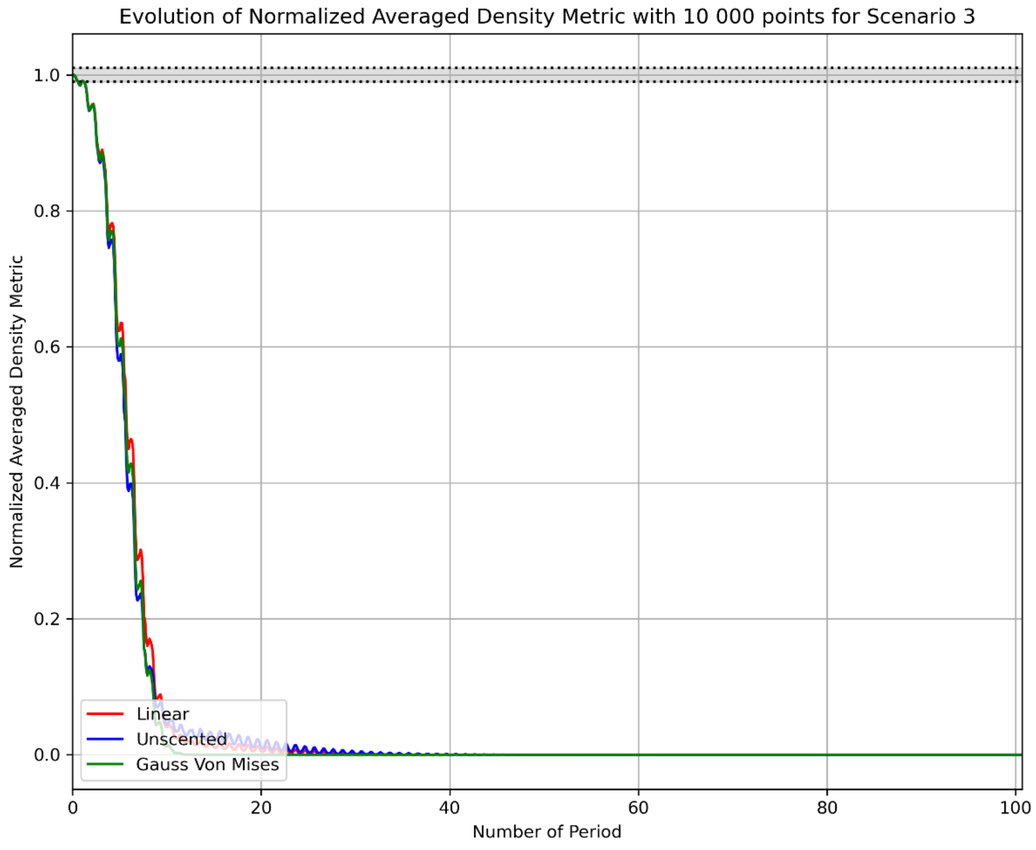


Credit: A comparative study of new non-linear uncertainty propagation methods for space surveillance

Scenario	Orbit Type	Orbit Accuracy	a(km)	e	i(°)	Ω(°)	ω(°)	M(°)	Timespan (days)
3	LEO	High	7136.6	0.00949	72.9	116.0	57.7	105.5	7

RESULTS

Scenario 3



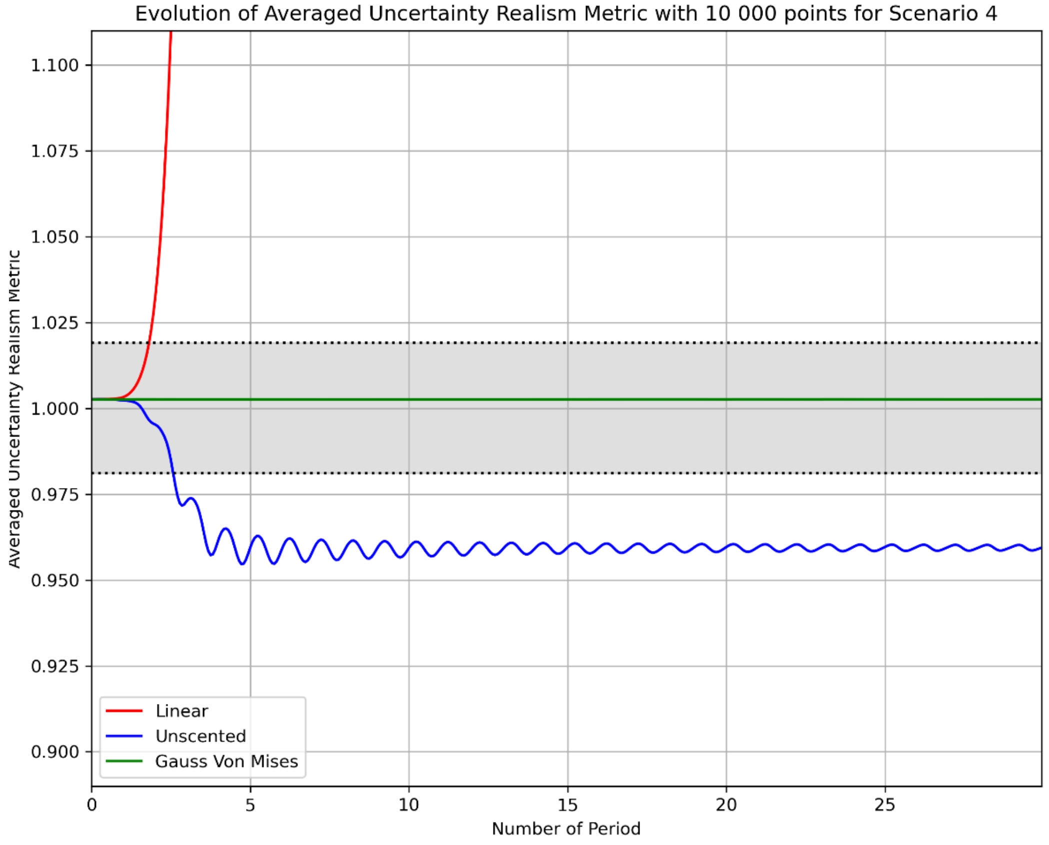
	NADM (orbit)
Linear	1.18
Unscented	1.18
Gauss Von Mises	1.18

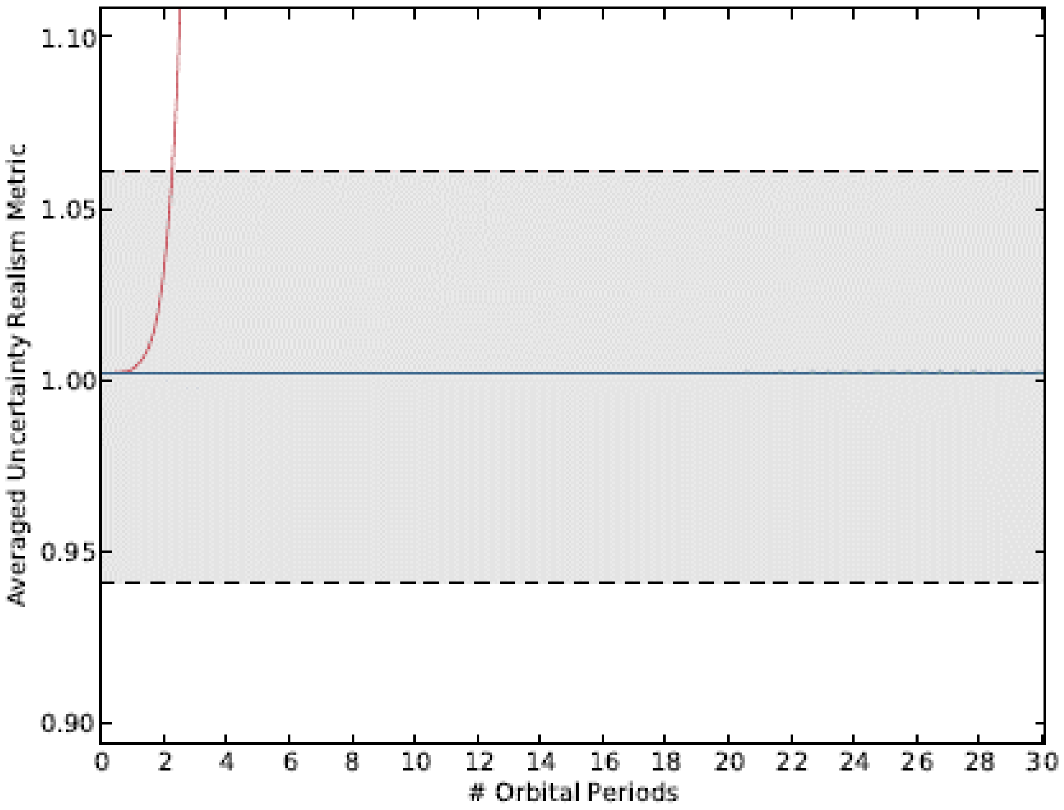
RESULTS

Scenario 4 – Linear

Scenario	Orbit Type	Orbit Accuracy	a(km)	e	i(°)	Ω(°)	ω(°)	M(°)	Timespan (days)
4	GEO	Medium	42164.1	0.0	0.0	0.0	0.0	0.0	30

Focus on the red curve





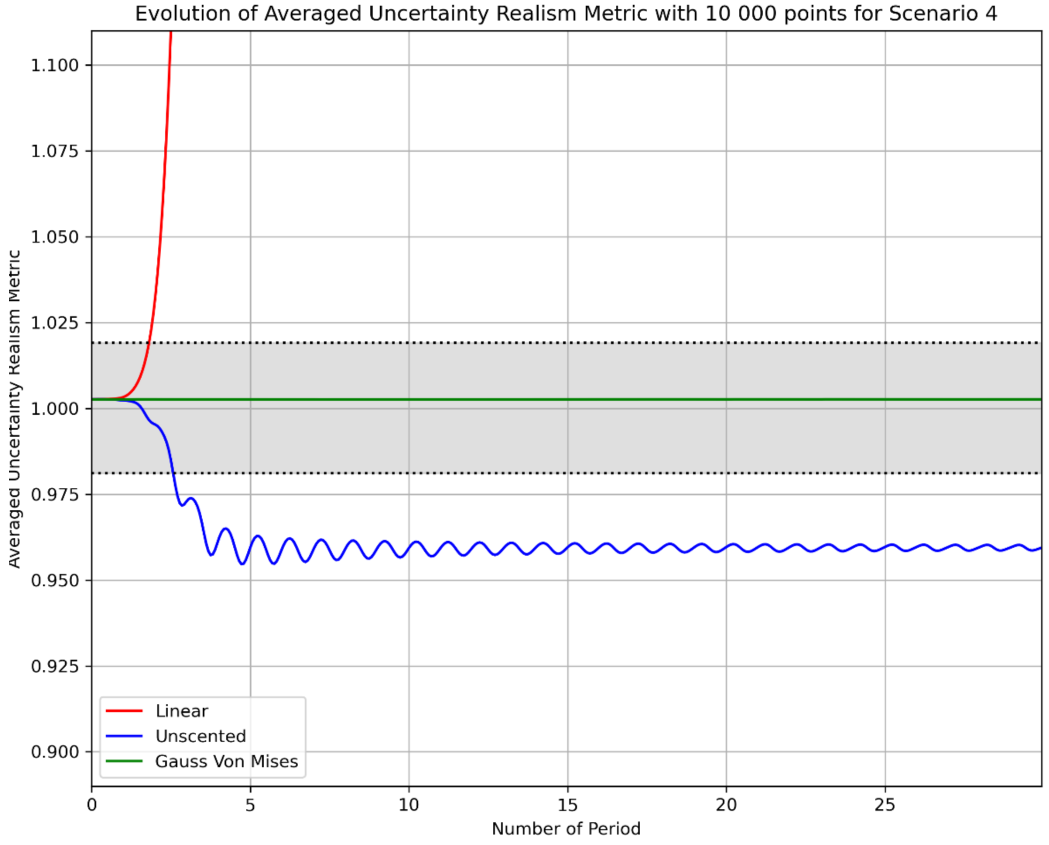
Credit: A comparative study of new non-linear uncertainty propagation methods for space surveillance

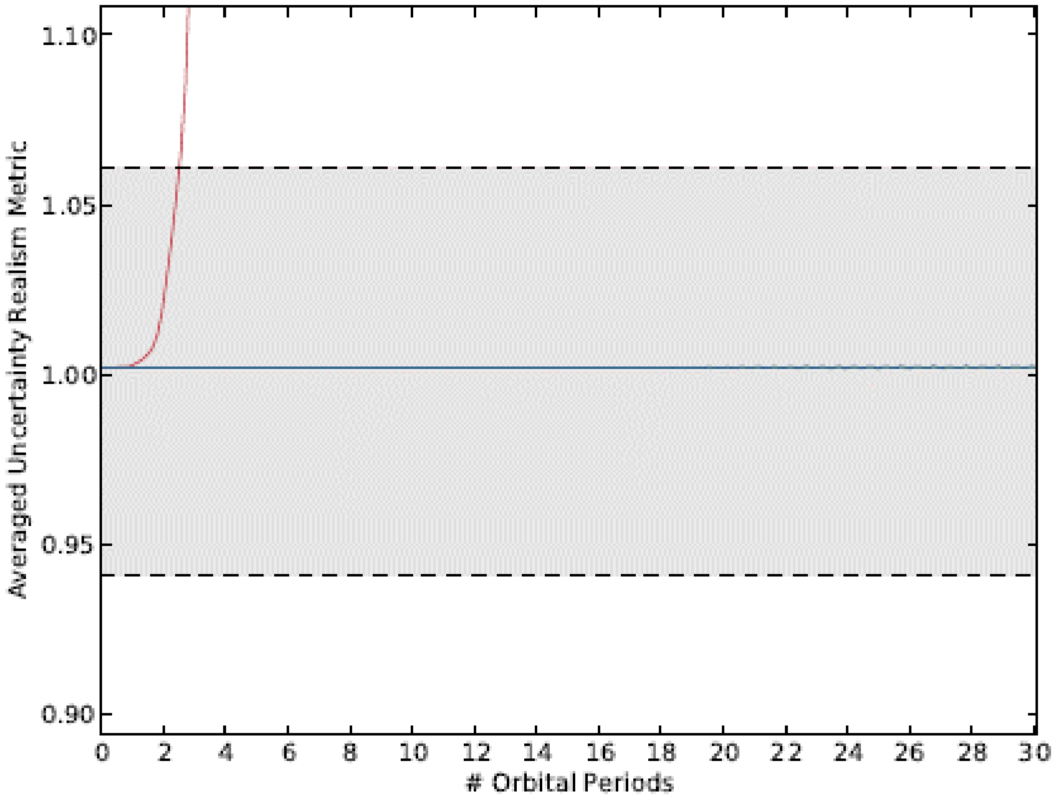
Scenario	Orbit Type	Orbit Accuracy	a(km)	e	i(°)	Ω(°)	ω(°)	M(°)	Timespan (days)
4	GEO	Medium	42164.1	0.0	0.0	0.0	0.0	0.0	30

RESULTS

Scenario 4 – Unscented

Focus on the red curve





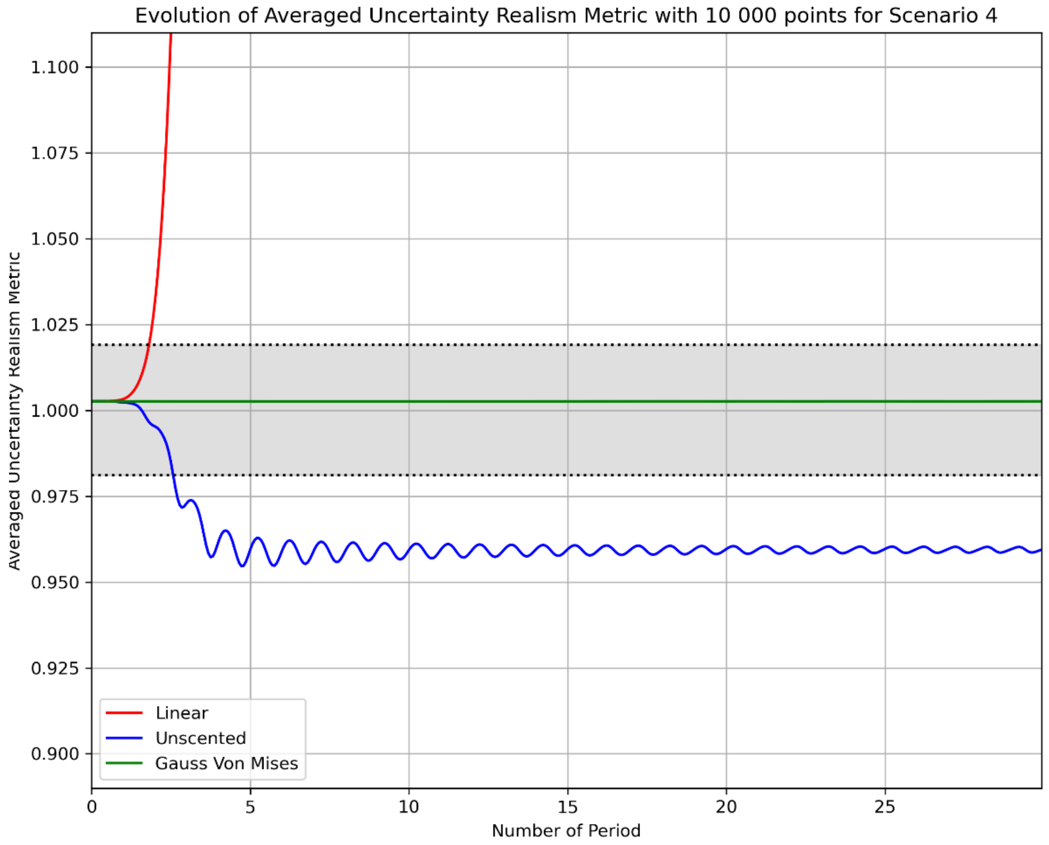
Credit: A comparative study of new non-linear uncertainty propagation methods for space surveillance

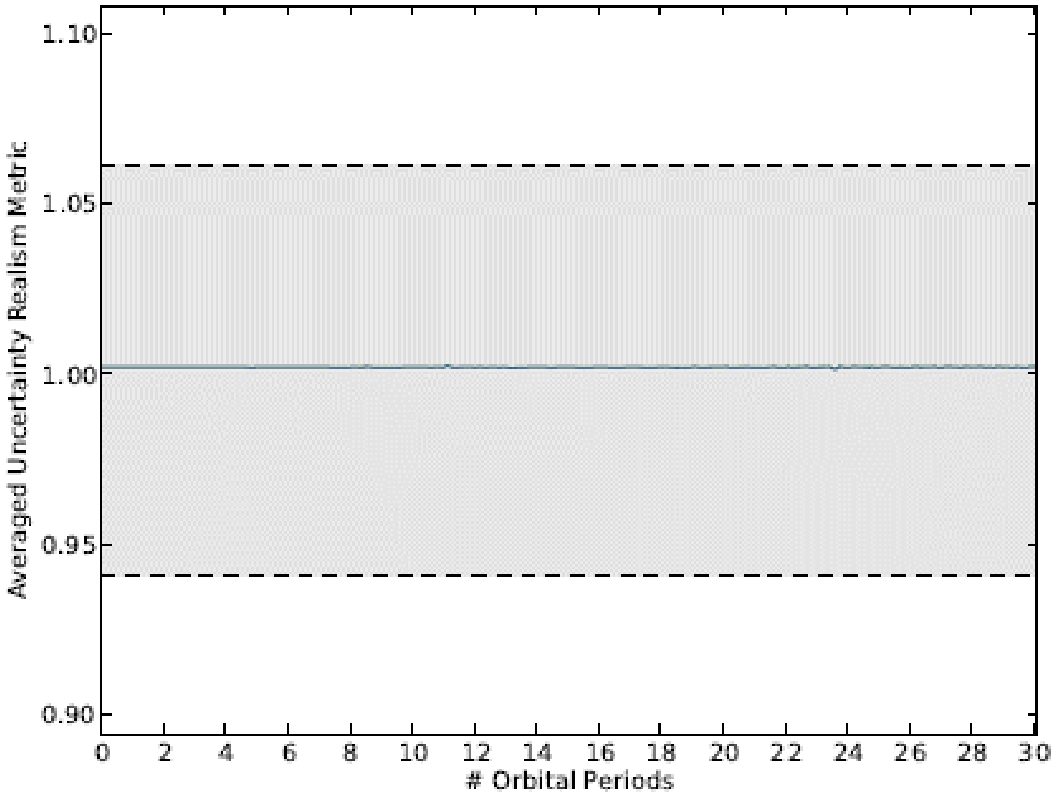
Scenario	Orbit Type	Orbit Accuracy	a(km)	e	i(°)	Ω(°)	ω(°)	M(°)	Timespan (days)
4	GEO	Medium	42164.1	0.0	0.0	0.0	0.0	0.0	30

RESULTS

Scenario 4 – Gauss Von Mises

Focus on the green curve



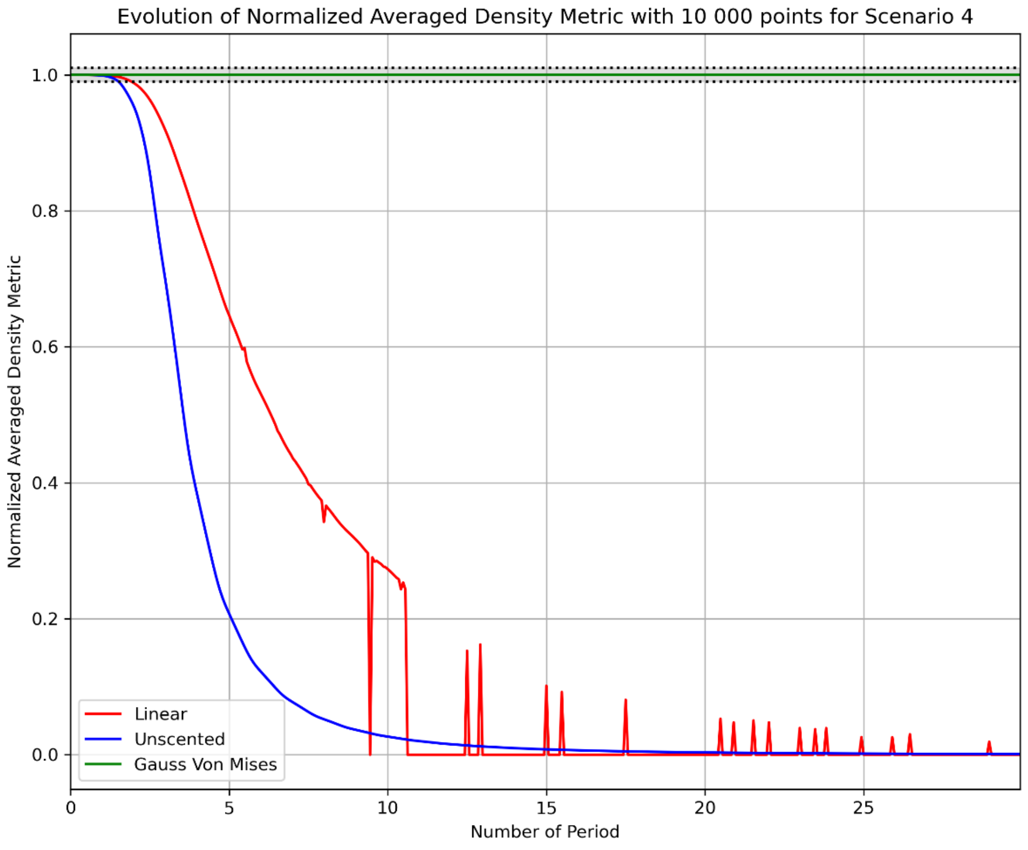


Credit: A comparative study of new non-linear uncertainty propagation methods for space surveillance

Scenario	Orbit Type	Orbit Accuracy	a(km)	e	i(°)	Ω(°)	ω(°)	M(°)	Timespan (days)
4	GEO	Medium	42164.1	0.0	0.0	0.0	0.0	0.0	30

RESULTS

Scenario 4



	NADM (orbit)
Linear	1.929
Unscented	1.483
Gauss Von Mises	> 30

RESULTS

Conclusion

	Scenario 1	Scenario 2	Scenario 3	Scenario 4	Scenario 5
Result	Conclusive	Conclusive	Inconclusive	Conclusive	Inconclusive

REFERENCES

- **Unscented covariance:**

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Aubrey B. Poore, Jeffrey M. Aristoff, and Joshua T. Horwood, “Covariance and Uncertainty Realism in Space Surveillance and Tracking”

David MIMOUN, *“Restitution d’orbite avec filtre de Kalman Unscented”*

- **Gauss Von Mises Distribution:**

Joshua T. Horwood, Aubrey B. Poore, *“Gauss von Mises Distribution for Improved Uncertainty Realism in Space Situational Awareness”*

Joshua T. Horwood, Aubrey B. Poore, *“Orbital State Uncertainty Realism”*

J. T. Horwood, J. M. Aristoff, N. Singh, and A. B. Poore, *“A comparative study of new non-linear uncertainty propagation methods for space surveillance,”* in Proceedings of the SPIE, Signal and Data Processing of Small Targets, Vol. 9092, Baltimore, MD, May 2014

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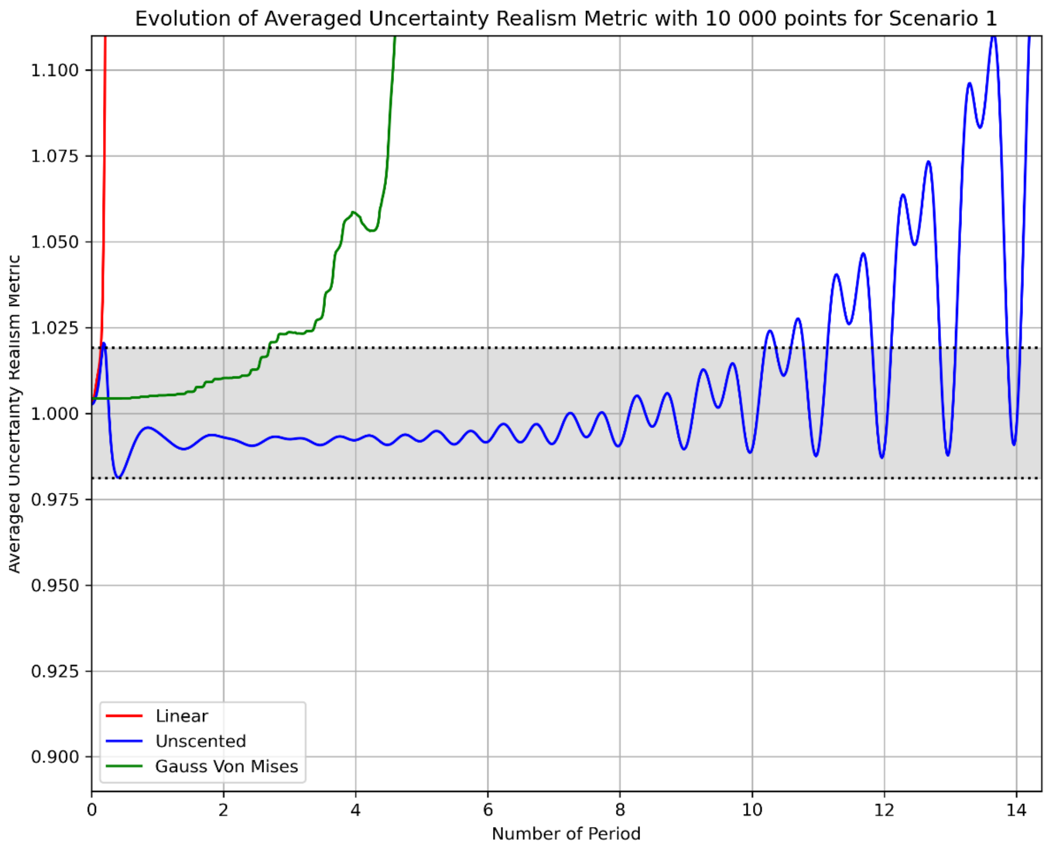
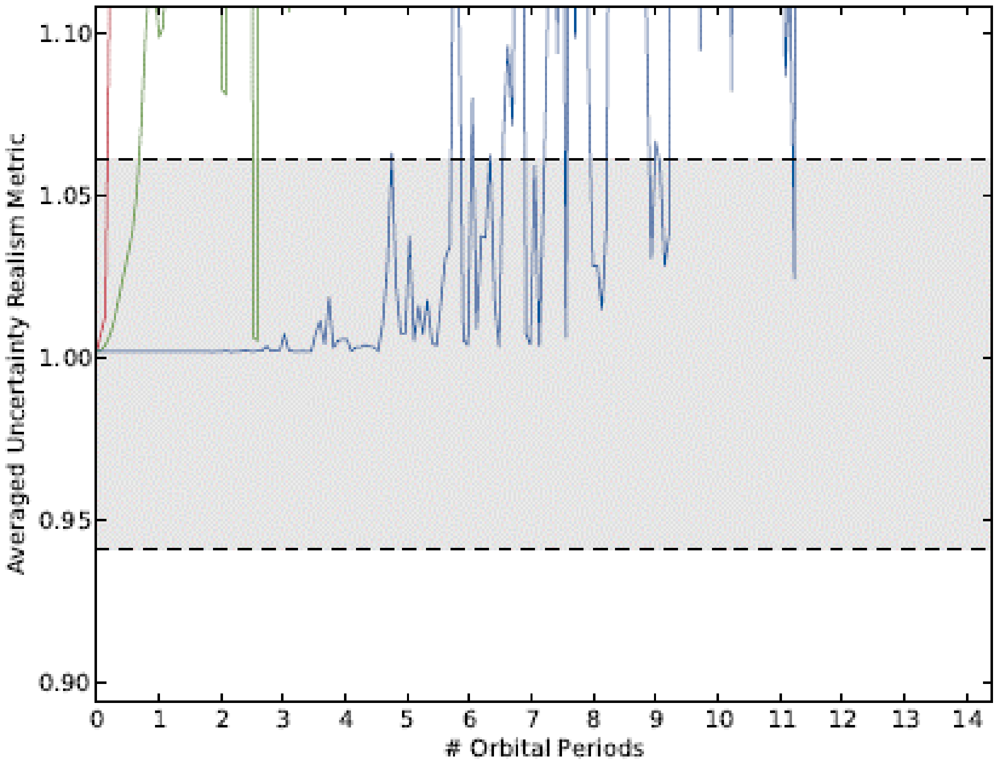
Scenario	Orbit Type	Orbit Accuracy	a(km)	e	i(°)	Ω(°)	ω(°)	M(°)	Timespan (days)
1	LEO	Low	7136.6	0.00949	72.9	116.0	57.7	105.5	1

APPENDICES

Scenario 1 – Linear

Focus on the red curve





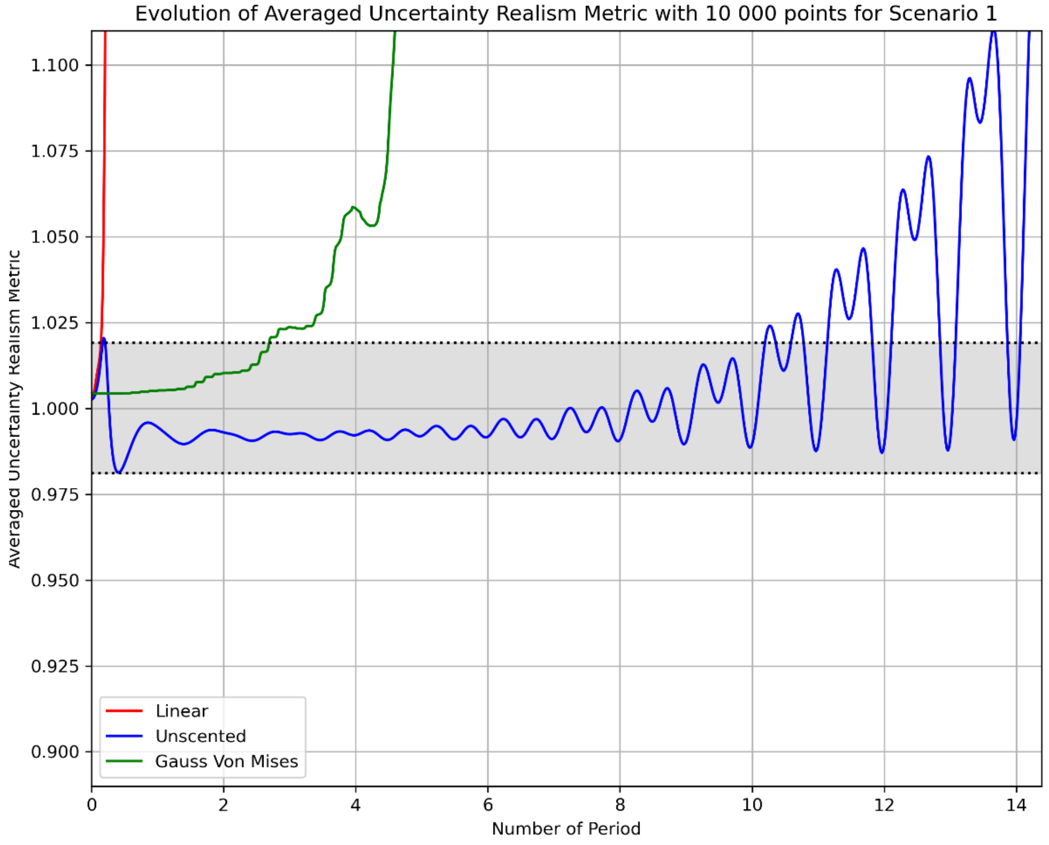
Credit: A comparative study of new non-linear uncertainty propagation methods for space surveillance

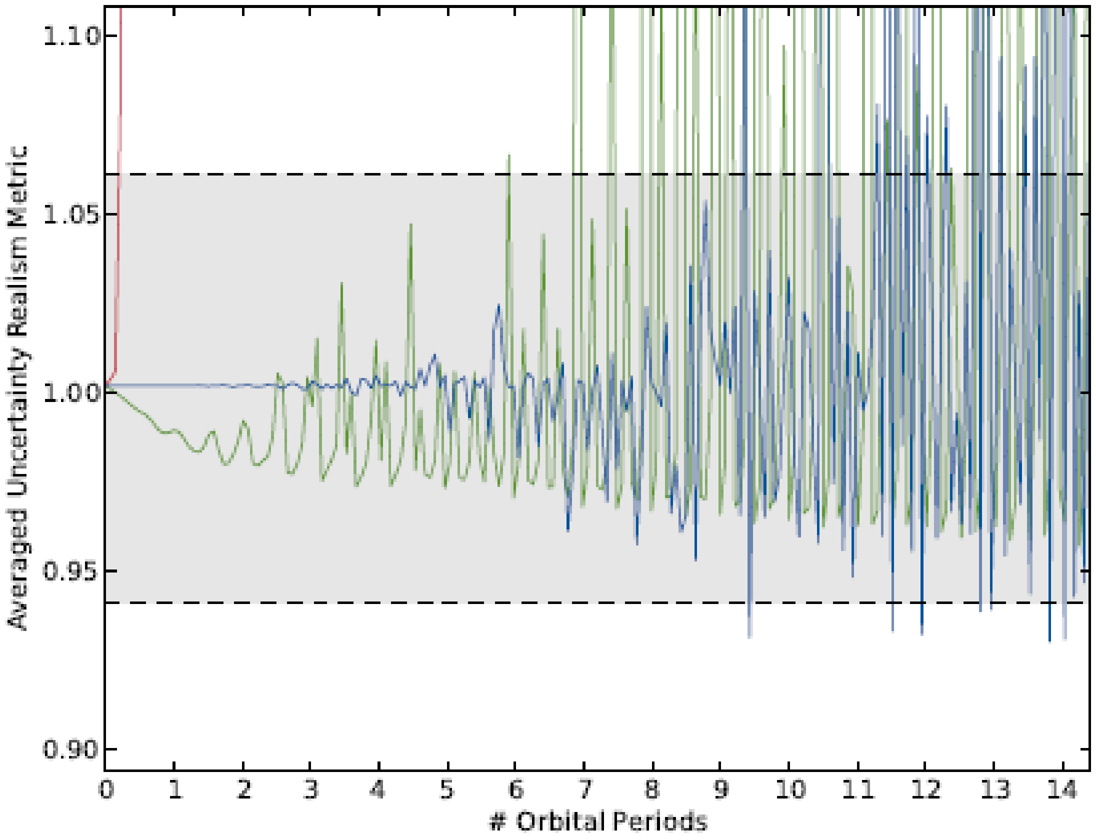
Scenario	Orbit Type	Orbit Accuracy	a(km)	e	i(°)	Ω(°)	ω(°)	M(°)	Timespan (days)
1	LEO	Low	7136.6	0.00949	72.9	116.0	57.7	105.5	1

APPENDICES

Scenario 1 – Unscented

Focus on the red curve





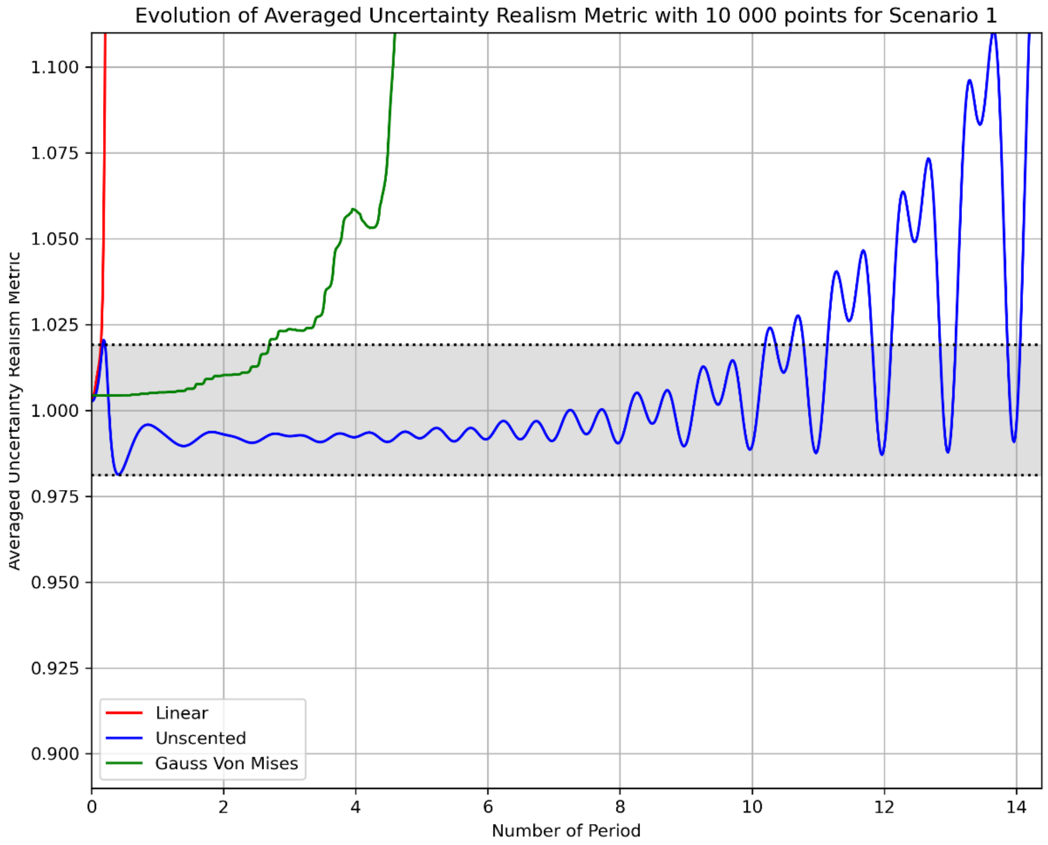
Credit: A comparative study of new non-linear uncertainty propagation methods for space surveillance

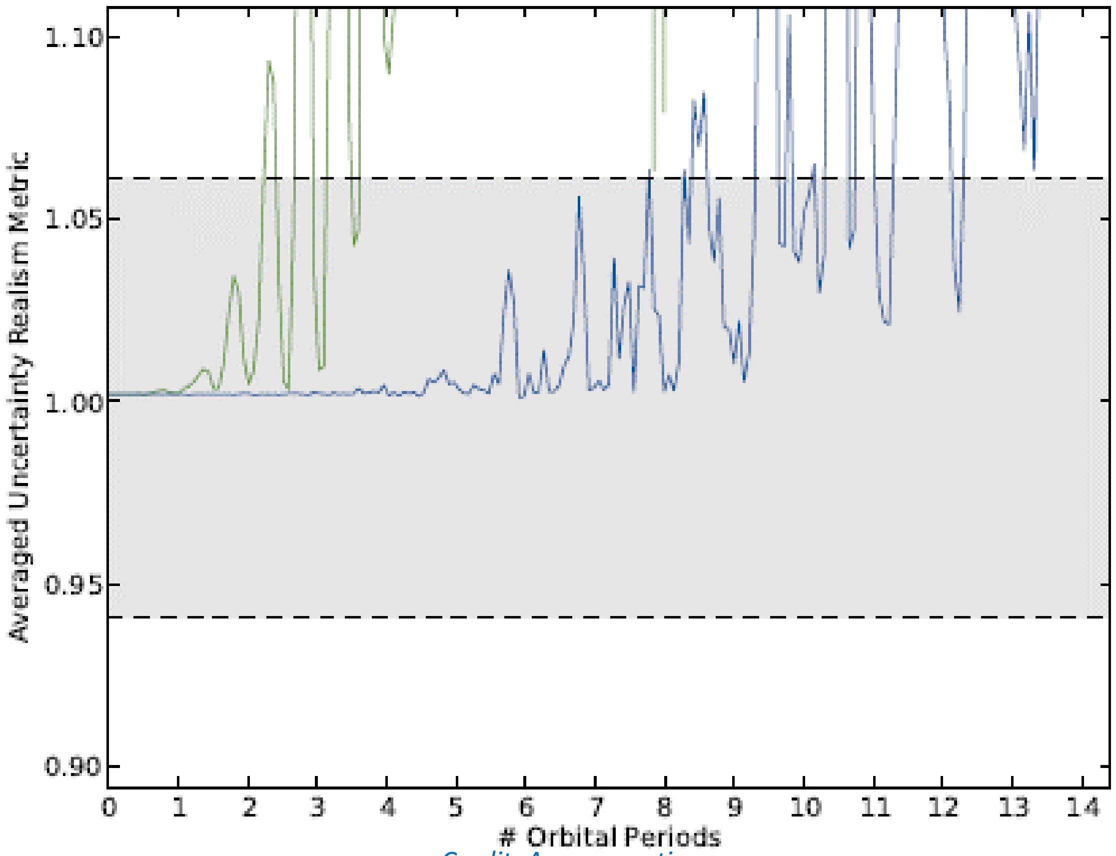
Scenario	Orbit Type	Orbit Accuracy	a(km)	e	i(°)	Ω(°)	ω(°)	M(°)	Timespan (days)
1	LEO	Low	7136.6	0.00949	72.9	116.0	57.7	105.5	1

APPENDICES

Scenario 1 – Gauss Von Mises

Focus on the green curve





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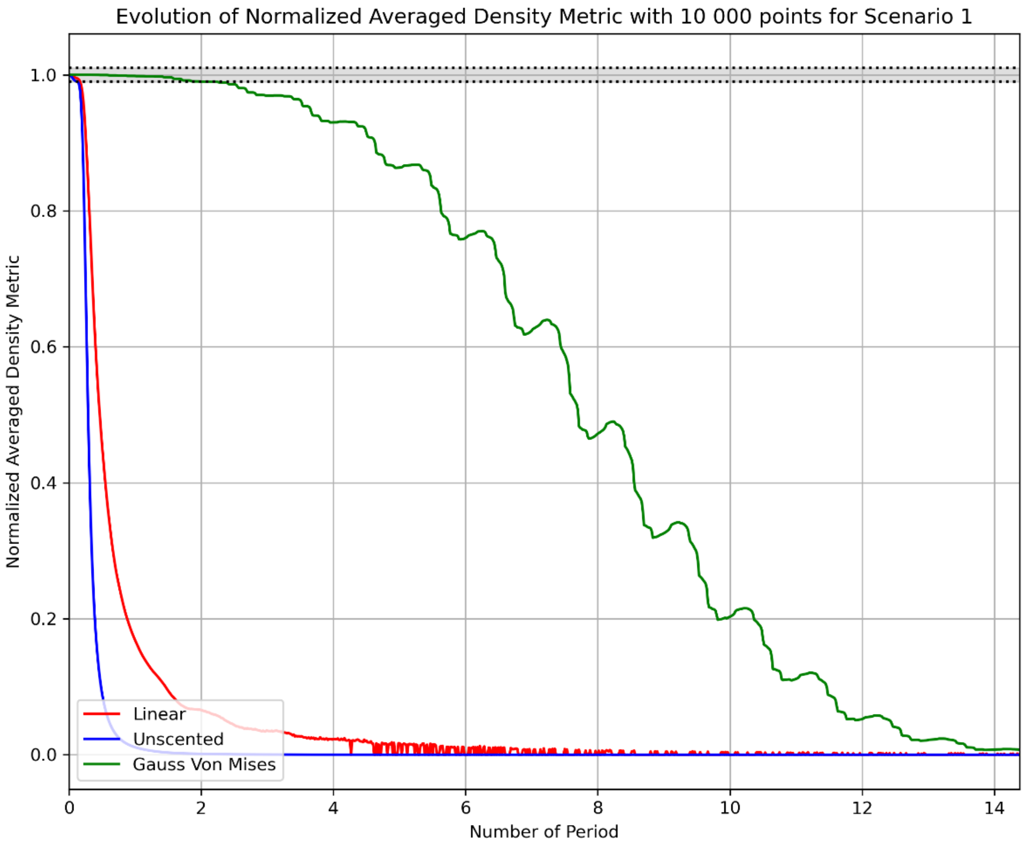
APPENDICES

Scenario 1

Scenario	Orbit Type	Orbit Accuracy	a(km)	e	i(°)	Ω(°)	ω(°)	M(°)	Timespan (days)
1	LEO	Low	7136.6	0.00949	72.9	116.0	57.7	105.5	1



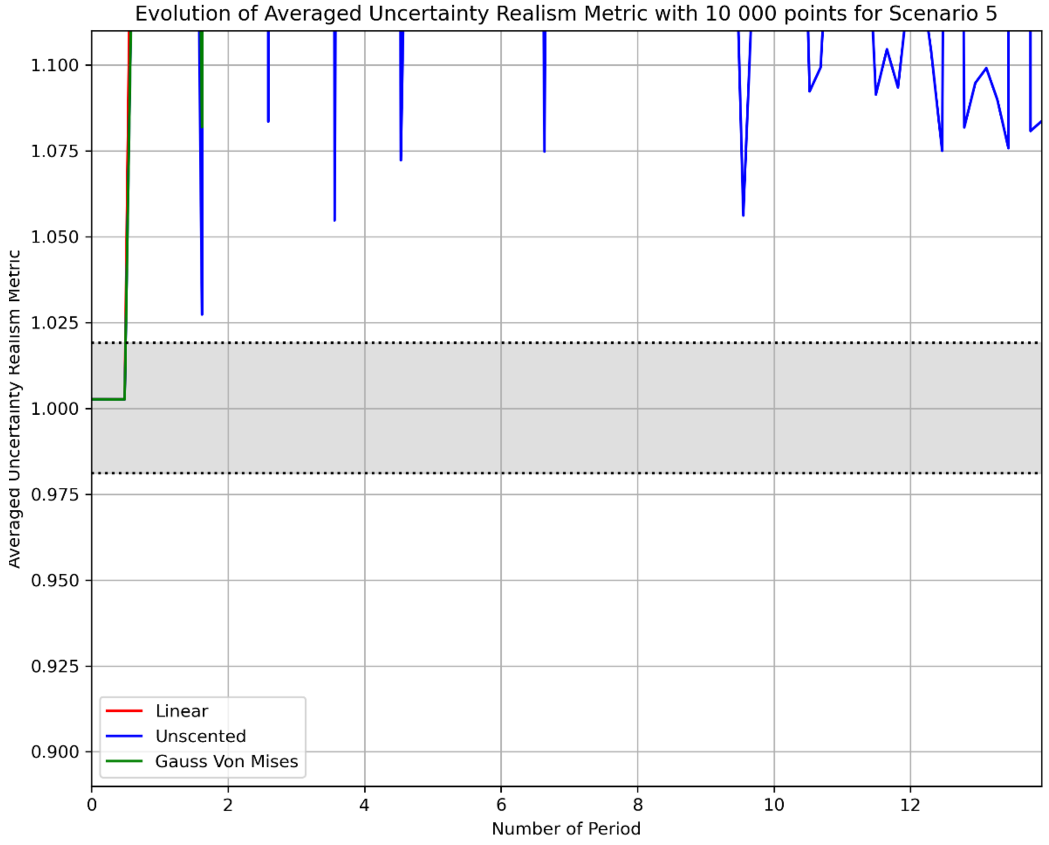
	NADM (orbit)
Linear	0.18
Unscented	0.142
Gauss Von Mises	2.226

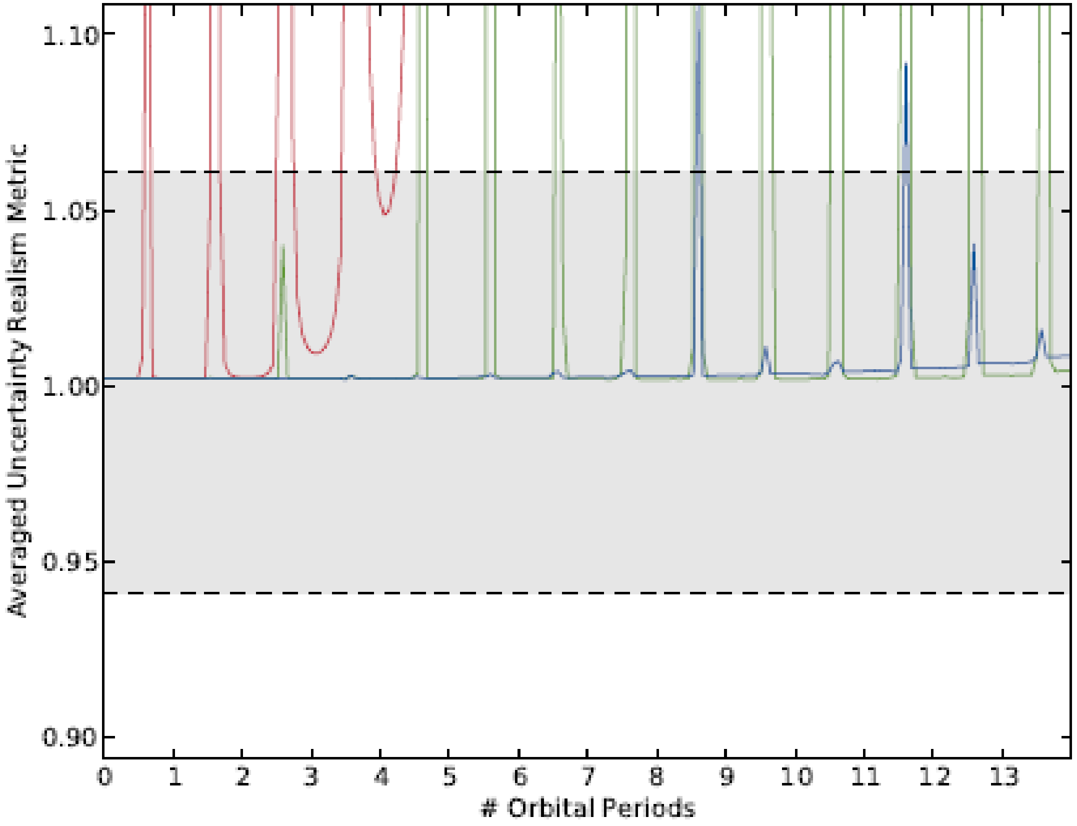
Scenario	Orbit Type	Orbit Accuracy	a(km)	e	i(°)	Ω(°)	ω(°)	M(°)	Timespan (days)
5	HEO	Medium	26628.1	0.742	63.4	120.0	0.0	144.0	7

APPENDICES

Scenario 5 – Linear

Focus on the red curve





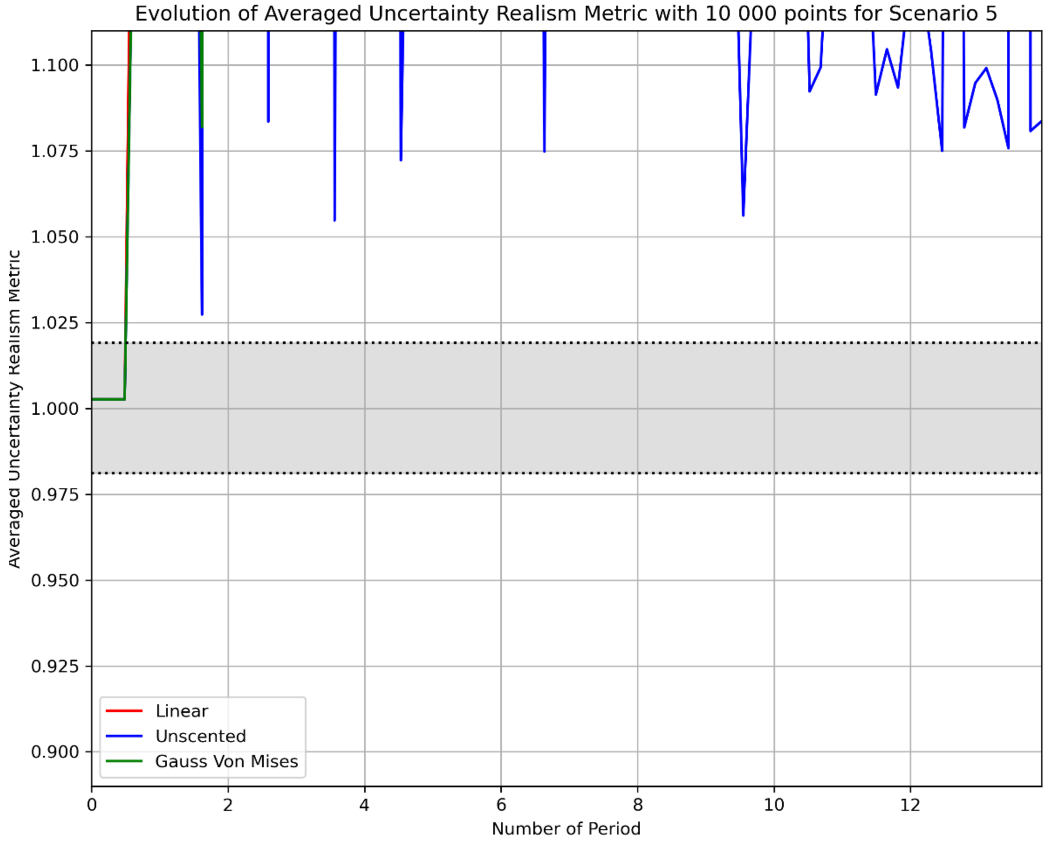
Credit: A comparative study of new non-linear uncertainty propagation methods for space surveillance

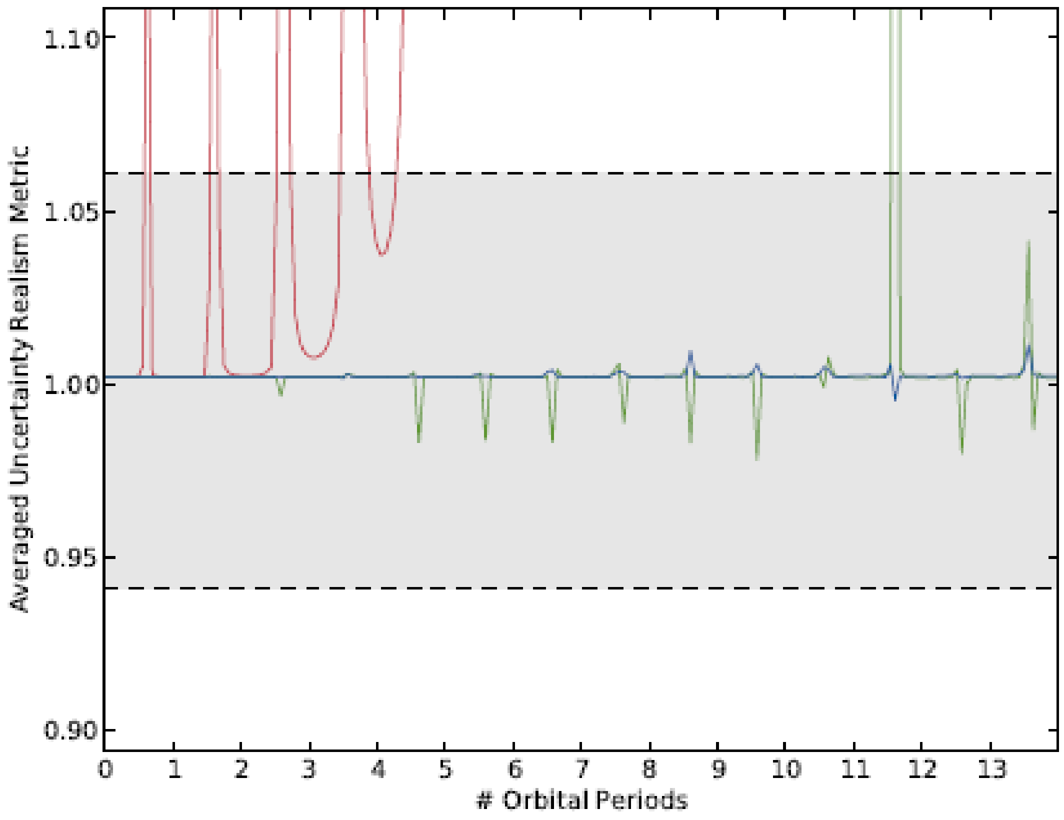
Scenario	Orbit Type	Orbit Accuracy	a(km)	e	i(°)	Ω(°)	ω(°)	M(°)	Timespan (days)
5	HEO	Medium	26628.1	0.742	63.4	120.0	0.0	144.0	7

APPENDICES

Scenario 5 – Unscented

Focus on the red curve





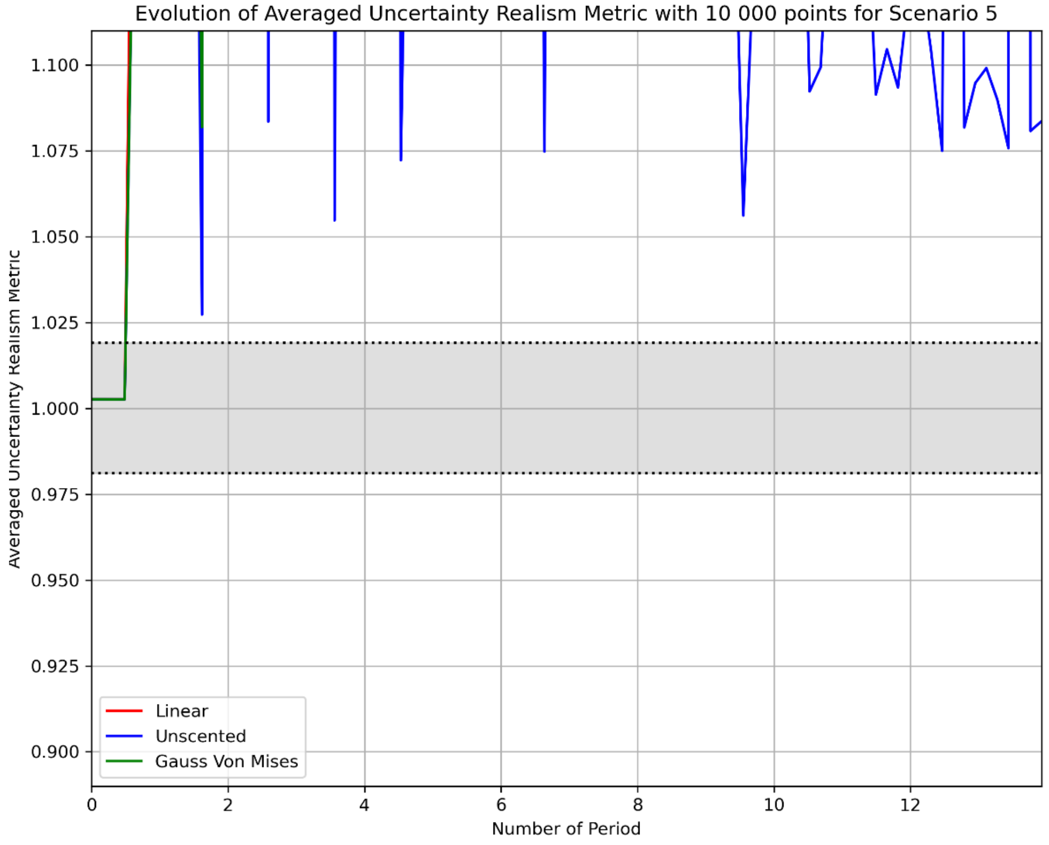
Credit: A comparative study of new non-linear uncertainty propagation methods for space surveillance

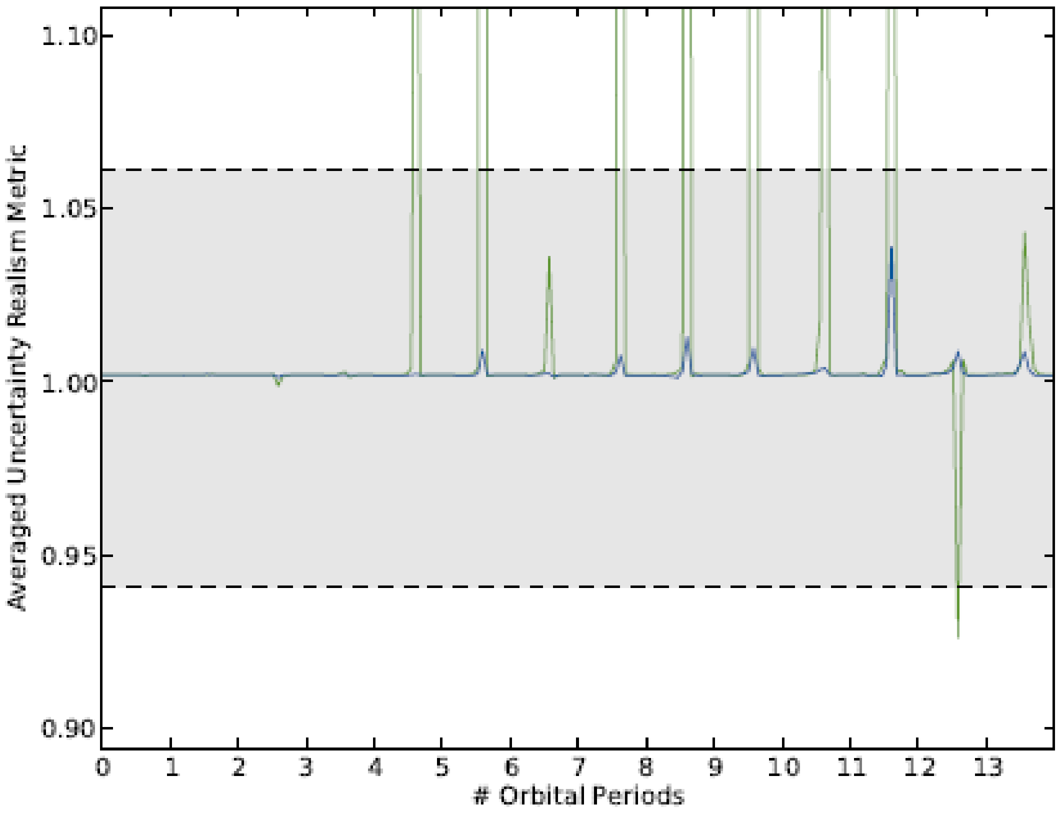
Scenario	Orbit Type	Orbit Accuracy	a(km)	e	i(°)	Ω(°)	ω(°)	M(°)	Timespan (days)
5	HEO	Medium	26628.1	0.742	63.4	120.0	0.0	144.0	7

APPENDICES

Scenario 5 – Gauss Von Mises

Focus on the green curve



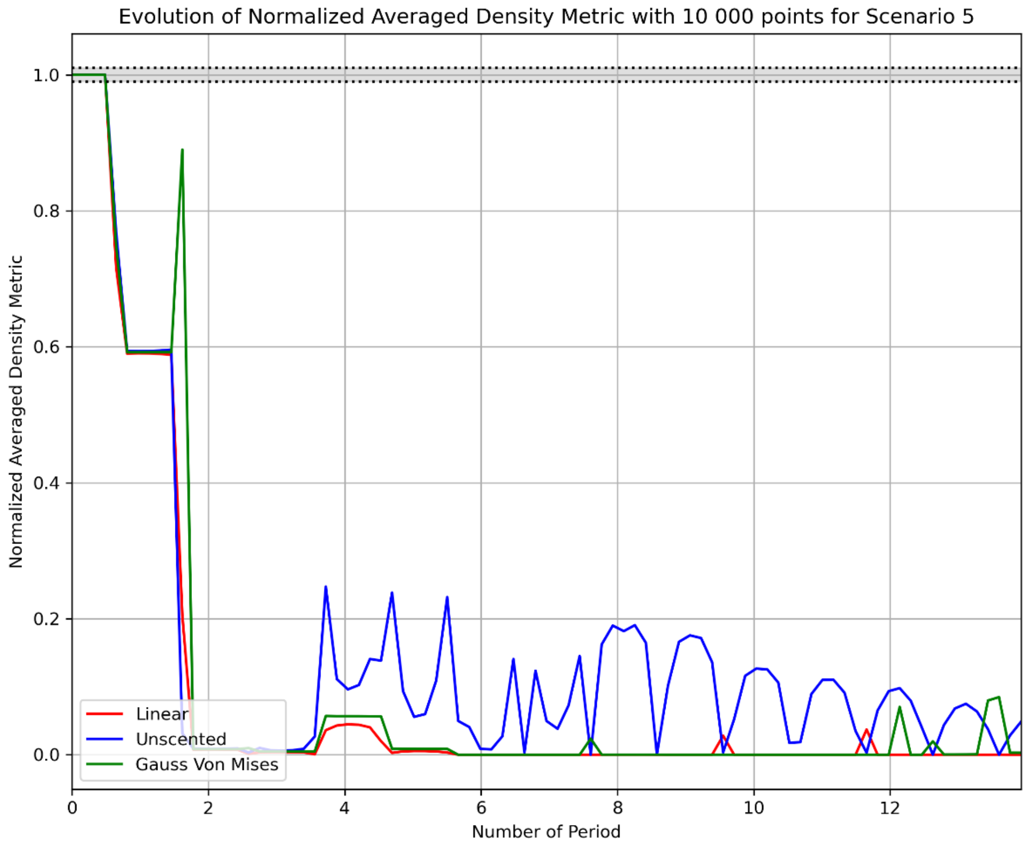


Credit: A comparative study of new non-linear uncertainty propagation methods for space surveillance

APPENDICES

Scenario 5

Scenario	Orbit Type	Orbit Accuracy	a(km)	e	i(°)	Ω(°)	ω(°)	M(°)	Timespan (days)
5	HEO	Medium	26628.1	0.742	63.4	120.0	0.0	144.0	7



	NADM (orbit)
Linear	0.493
Unscented	0.493
Gauss Von Mises	0.493